On the reductive Borel-Serre compactification, II:
Excentric quotients and least common modifications

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Abstract. Let X be a locally symmetric variety, i.e., the quotient of a bounded symmetric domain by a (say) neat arithmetically-defined group of isometries. Let $\overline{X}^{\mathrm{exc}}$ and $X^{\mathrm{tor,exc}}$ denote its excentric Borel-Serre and toroidal compactifications respectively. We determine their least common modification and use it to prove a conjecture of Goresky and Tai concerning canonical extensions of homogeneous vector bundles. In the process, we see that $\overline{X}^{\mathrm{exc}}$ and $X^{\mathrm{tor,exc}}$ are homotopy equivalent.

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Introduction

This article is the continuation of [Z4] in the direction it left open, namely the Goresky-Tai conjecture [GT: 9.5] (Conjecture A in [Z4]). The conjecture concerns two very different classes of compactifications of a locally symmetric variety X, and the corresponding notions of canonical extension of homogeneous vector bundles $\mathcal{E}$ on X. One class, the good toroidal compactifications X^{tor} [AMRT], comes from X as an algebraic variety, with X^{tor} a smooth complex projective variety. The other, the Borel-Serre compactification $\overline{X}$ [BS] (a manifold-with-corners) and its reductive quotient $\overline{X}^{\mathrm{red}}$ (a real stratified space; see [Z4:§§1,5]), being defined also for non-Hermitian X, are more general. The Borel-Serre spaces have boundary strata of odd real-codimension, so are quite far from being complex-analytic spaces. Nonetheless, one can view Conjecture A as saying that $\overline{X}^{\mathrm{red}}$ is more fundamental than X^{tor}, at least as far as homogeneous vector bundles are concerned: if $\mathcal{E}^{\mathrm{tor}}$ and $\overline{\mathcal{E}}^{\mathrm{red}}$ denote the respective bundle extensions of $\mathcal{E}$ to X^{tor} and $\overline{X}^{\mathrm{red}}$, then for certain continuous mappings $h : X^{\mathrm{tor}} \to \overline{X}^{\mathrm{red}}$ (described below), one has $\mathcal{E}^{\mathrm{tor}} \simeq h^*\overline{\mathcal{E}}^{\mathrm{red}}$. In other words, $\mathcal{E}^{\mathrm{tor}}$ is determined as a topological vector bundle by $\overline{\mathcal{E}}^{\mathrm{red}}$; their respective Chern classes are correspondingly related.

A problem in dealing with Conjecture A is that X^{tor} and $\overline{X}^{\mathrm{red}}$ are generally so different. Here, I am thinking *beyond* the result of Lizhen Ji [J] (in essence, the conjecture from [HZ2:(1.5.8)] revised), which implies that the greatest common quotient of X^{tor} and $\overline{X}^{\mathrm{red}}$ is the obvious one, namely the Baily-Borel compactification X^* (see (2.2.18) and (2.3.5)). (Each of X^{tor} and $\overline{X}$ can be viewed as a resolution of X^*, in the senses of complex algebraic varieties and real stratified spaces, resp.) In [HZ2:§1], we introduced two auxiliary compactifications of X which we now denote $X^{\mathrm{tor,exc}}$ and $\overline{X}^{\mathrm{exc}}$, and call the *excentric* toroidal and Borel-Serre compactifications respectively (see our Section 2). They are straightforward quotients of X^{tor} and $\overline{X}$ respectively, with $\overline{X}^{\mathrm{exc}}$ mapping onto $\overline{X}^{\mathrm{red}}$. The two excentric compactifications could be seen to have much in common (see, e.g., (2.3.11)). It was a naive suspicion at the time that they might be isomorphic as compactifications of X; it was important to realize that in general they are not.

The goal in [GT] was to determine the least common modification (language as in [HZ2:(1.1.4)]) of $\overline{X}^{\mathrm{red}}$ and X^{tor}, denoted $\mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}})$. They showed that the canonical mapping $\mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) \to X^{\mathrm{tor}}$ is a homotopy equivalence. Then

homotopy inverses are used to yield the mappings h above, via the composite

$$X^{\mathrm{tor}} \to \mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) \to \overline{X}^{\mathrm{red}}.$$

In this article, we determine that $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}$ is a homotopy equivalence (Theorem 3.1.1), for more or less the same reason. However, our argument shows that it is a natural consequence of the calculation of the least common modification of dual compactifications of a simplicial cone (1.2) (see also [HZ1: 2.3]). We thereby obtain a homotopy class of mappings $k : X^{\mathrm{tor,exc}} \to \overline{X}^{\mathrm{exc}}$. We recover the LCM determination from [GT] by considerations of *LCM-basechange* (1.1.8), the name we give to the determination for three compactifications X_1, X_2, and X_3, with X_3 a quotient of X_2, whether the canonical embedding

$$\mathrm{LCM}(X_1, X_2) \hookrightarrow \mathrm{LCM}(X_1, X_3) \times_{X_3} X_2$$

is an equality. On the other hand, our theorem does not seem to follow directly from [GT].

The preceding is made more significant by the fact that there is a canonical extension $\mathcal{E}^{\mathrm{tor,exc}}$ of the homogeneous vector bundle $\mathcal{E}$ to $X^{\mathrm{tor,exc}}$ that pulls back to $\mathcal{E}^{\mathrm{tor}}$ via $X^{\mathrm{tor}} \to X^{\mathrm{tor,exc}}$ (4.3), and likewise $\overline{\mathcal{E}}^{\mathrm{exc}}$ on $\overline{X}^{\mathrm{exc}}$ that is the pullback of $\overline{\mathcal{E}}^{\mathrm{red}}$ via $\overline{X}^{\mathrm{exc}} \to \overline{X}^{\mathrm{red}}$. We can then formulate the analogue of Conjecture A (which we tentatively label Conjecture A′: $k^* \overline{\mathcal{E}}^{\mathrm{exc}} \simeq \mathcal{E}^{\mathrm{tor,exc}}$). It is not hard to see that Conjecture A is a consequence of Conjecture A′, but the latter is talking about the pair of excentric compactifications, which more resemble each other. We give a proof of Conjecture A′ in the last section of this article.

The organization of this article is as follows. In Section 1, we treat the compactifications of a space X as a category in (1.1), in which taking the LCM is a bifunctor. Particularly important is the discussion of when LCM-basechange occurs. In (1.2), we make the critical calculation of the LCM of dual compactifications of a simplicial cone. This is followed in (1.3) by a treatment of the somewhat elusive notion that we call *diagonality*, where the commutation of LCM and discrete quotients is analyzed.

Section 2 contains a description of the essential features of the various classes of compactifications of a locally symmetric variety: Baily-Borel (2.1), toroidal (2.2), and Borel-Serre (2.3). The excentric quotients are recalled in (2.2.18) and (2.3.5). In an appendix, we introduce some additional Borel-Serre quotients (called *hybrid*

compactifications), lying between $\overline{X}^{\mathrm{exc}}$ and $\overline{X}^{\mathrm{red}}$, that are inspired by the determination of the greatest common quotient of $\overline{X}$ and X^{tor} in [J].

Section 3 is devoted to the proof of our Theorem 3.1.1, which asserts that $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}$ is a homotopy equivalence. We use the outcome of (1.2) to give in (3.2) an approximation of the proof, revising it at the boundary in (3.3). In Sections (3.4) and (3.5), we finish the proof and determine consequences that follow by LCM-basechange. We emphasize the substantial conclusion given in Corollary 3.5.11: $\overline{X}^{\mathrm{exc}}$ *and* $X^{\mathrm{tor,exc}}$ *are homotopy equivalent.*

Finally, in Section 4, we carry out the toroidal construction of [AMRT] on (the total space of) the vector bundle $\mathcal{E}$, yielding a vector bundle $\mathcal{E}^{\mathrm{tor}}$ on X^{tor}. We show that $\mathcal{E}^{\mathrm{tor}}$ is the canonical extension of $\mathcal{E}$ in the sense of [Mu] in (4.3). (This is analogous to what was done in [Z4] to yield $\overline{\mathcal{E}}^{\mathrm{red}}$.) We then descend this bundle to $\mathcal{E}^{\mathrm{tor,exc}} \to X^{\mathrm{tor,exc}}$. In (4.4), we verify that Conjecture A′ implies the conjecture of Goresky-Tai (Conjecture A). We complete the task by verifying Conjecture A′ in (4.5).

I want to thank Mark Goresky and Lizhen Ji for helpful correspondence and discussions. The referee is to be commended for his or her thorough job of reading the manuscript, and also for wisely insisting on a substantial revamping of the exposition. To my surprise, it was pointed out by the referee (correctly!) that the verbal description in the first line of [HZ1: p.262] is a bit garbled; the correct assertion was nearby, though, and it is stated correctly (in slightly different notation) here in (2.2).

Apology. It is only a little unnatural that this article is appearing well after its sequel [Z5], which was written for a special volume. The latter is largely independent, referring only to Corollary 3.5.11 for a conditional assertion. There is also reference to Corollary 3.5.11 in [Z6: Prop. 2].

Comments on [Z4]*.* i) **Erratum:** The quantity δ in [Z4:(3.1.4)] should be described as, and taken to be, the *sum* of the positive $\mathbb{Q}$-roots, not the half-sum (the "half" appears as $\frac{1}{p}$ when $p = 2$). Subsequent statements involving δ are correct as written.

ii) About the time [Z4] appeared, R. Mazzeo asked a familiar question: "Why L^p-cohomology for $p \neq 2$?" I think that the article [Z4] provides a good answer. It is almost certain that I first tried to calculate the L^∞-cohomology quickly, finding it to be infinite-dimensional. Use of large finite p offers a perturbation away from that difficulty, allowing for our topological interpretation of the L^p-cohomology.

Changes in notation from [Z4]. The simultaneous treatment of Borel-Serre and toroidal compactifications taxes one's alphabetical resources, as was seen already in [HZ2]. Note in particular the following changes of notation:

i) The unipotent radical of a parabolic subgroup P (formerly U_P) is now denoted W_P; the symbol U_P is used here for the *center* of the unipotent radical. This is as in [AMRT].

ii) Symmetric spaces (formerly X) are now denoted D; the symbol X is now used for the arithmetic quotient $\Gamma \backslash D$, which was denoted M in [Z4].

iii) The reductive Borel-Serre compactification of X is denoted here $\overline{X}^{\mathrm{red}}$ (formerly M^{RBS} of M). The "bar" (overline) always indicates a construction of Borel-Serre type.

1. The category of compactifications of a space

The main purpose of this section is to recall (cf. [HZ2,§1]) the notion of the least common modification of two compactifications of the same topological space, and to develop properties that this notion has when a third compactification is invoked.

(1.1) Fundamentals. We begin with some basic terminology. Let X be a locally compact Hausdorff space (non-compact) with a countable base for its topology. One defines a category $\mathfrak{Cp}(X)$, the *compactifications of X*, as follows. The objects are pairs (Y, ι), with Y a compact Hausdorff space, and $\iota : X \to Y$ a dense open embedding. We will write just Y when ι is understood, and view X as a subset of Y. The *interior* of Y is then understood to be X; the *boundary ∂Y* of Y (qua compactification of X) is the complement of X in Y, a closed subset of Y. To eliminate pathology, we will assume throughout this article that we allow only spaces Y (thus also X) whose topologies have countable neighborhood bases. This holds in particular whenever Y, hence also X, is metrizable.

A morphism in $\mathfrak{Cp}(X)$, called a *morphism of compactifications* of X, from (Y_1, ι_1) to (Y_2, ι_2), is a commutative triangle of continuous mappings:

$$
\begin{array}{ccc}
X & \xrightarrow{\iota_1} & Y_1 \\
& \searrow{\scriptstyle \iota_2} & \downarrow \\
& & Y_2
\end{array}
$$

Note that there is at most one morphism from Y_1 to Y_2, as X is dense in Y_1, and that morphisms in $\mathfrak{Cp}(X)$ are surjective by compactness. There is such a morphism

if and only if Y_2 is a topological quotient of Y_1 at its boundary (so ∂Y_1 maps onto ∂Y_2) . If so, one also says that Y_1 is a *modification of* Y_2.

The set of compactifications of X can be seen to form a (non-distributive) lattice. Specifically, for Y_1 and Y_2 as above, one takes $Y_1 \vee Y_2$ to be the closure of X (equivalently, any dense subset of X) under the diagonal embedding in $Y_1 \times Y_2$. This is the *least common modification* of the two compactifications, and is also denoted $\mathrm{LCM}(Y_1, Y_2)$ or $\mathrm{LCM}_X(Y_1, Y_2)$; it admits canonical morphisms to Y_1 and Y_2. The notion passes to the set of homeomorphism classes of compactifications. Similarly, one takes $Y_1 \wedge Y_2$ to be the *greatest common quotient* $\mathrm{GCQ}(Y_1, Y_2)$ of Y_1 and Y_2, which can be realized as the inverse limit of all common quotients of Y_1 and Y_2 (the set of such is non-empty, for the one-point compactification of X is always a quotient of both). It receives canonical morphisms from Y_1 and Y_2.

Lemma 1.1.1. *In* $\mathfrak{Cp}(X)$, *the following three statements are equivalent:*

i) There is a morphism $Y_1 \to Y_2$.

ii) Via the natural projection, $\mathrm{LCM}(Y_1, Y_2) \simeq Y_1$.

iii) Via the quotient mapping, $\mathrm{GCQ}(Y_1, Y_2) \simeq Y_2$. $\square$

It is useful to introduce the category $\mathfrak{PCp}(X)$ of partial compactifications of X, which contains $\mathfrak{Cp}(X)$ as a full subcategory. A *partial compactification* of X is a space Y (not necessarily compact) containing X as a dense open subset. In particular, X itself is an object in $\mathfrak{PCp}(X)$. The notions of boundary, morphism and LCM can be extended verbatim to $\mathfrak{PCp}(X)$, though morphisms need not be surjective (thus are not necessarily quotient mappings). Moreover, $\mathrm{GCQ}(Y_1, Y_2)$ need not be defined, for Y_1 and Y_2 may have no common quotients at all. Nonetheless, the following version of Lemma 1.1.1 holds in $\mathfrak{PCp}(X)$:

Lemma 1.1.1.1. *Let* $Y_1, Y_2 \in \mathfrak{PCp}(X)$.

i) There is a morphism $Y_1 \to Y_2$ *if and only if, via the natural projection,* $\mathrm{LCM}(Y_1, Y_2) \simeq Y_1$.

ii) There is a surjective *morphism* $Y_1 \to Y_2$ *if and only if* $\mathrm{GCQ}(Y_1, Y_2)$ *exists and is isomorphic to* Y_2 *via the natural mapping,* $Y_2 \to \mathrm{GCQ}(Y_1, Y_2)$. $\square$

(1.1.1.2) *Example.* Let X be the interval $(0, 1)$, $Y_1 = (0, 1]$ and $Y_2 = [0, 1)$. One has $\mathrm{LCM}(Y_1, Y_2) \simeq X$. In particular, the canonical morphisms, $\mathrm{LCM}(Y_1, Y_2) \to Y_i$ ($i = 1, 2$), are not surjective. Also, $\mathrm{GCQ}(Y_1, Y_2)$ is not defined.

In the sequel, we will concern ourselves only with the LCM. Also, until stated otherwise, assertions are given for $\mathfrak{PCp}(X)$. The following is elementary:

Lemma 1.1.2. *In* $\mathfrak{PCp}(X)$, *LCM*(Y_1, Y_2) *is equal to*

$$\{(y_1, y_2) \in Y_1 \times Y_2 \mid \exists \text{ a sequence } \{x_j\} \text{ in } X \text{ with } \iota_1(x_j) \to y_1, \iota_2(x_j) \to y_2\}. \quad \square$$

(1.1.3) *Remark.* When we want to draw more attention to the role of the boundaries ∂_1 and ∂_2 of Y_1 and Y_2 respectively, we renotate $\partial\mathrm{LCM}(Y_1, Y_2)$ as $\mathrm{LCMb}(\partial_1, \partial_2)$ ("b" as in "boundary"); in general, whenever we write $\mathrm{LCMb}(\partial_1, \partial_2)$, X and the way that ∂_1 and ∂_2 are attached to X are understood to be given information.

It is easy to see:

Proposition 1.1.4. *Let* Y_1, Y_2 *and* Z *be partial compactifications of* X, *such that* Y_1 *and* Y_2 *admit morphisms to* Z. *Then* $LCM(Y_1, Y_2)$ *maps to* Z, *and the following diagram commutes*

$$
\begin{array}{ccc}
\mathrm{LCM}(Y_1, Y_2) & \longrightarrow & Y_2 \\
\downarrow & & \downarrow{\scriptstyle \varphi_2} \\
Y_1 & \xrightarrow{\ \varphi_1\ } & Z
\end{array}
$$

In other words, $\mathrm{LCM}(Y_1, Y_2) \subseteq Y_1 \times_Z Y_2 \subset Y_1 \times Y_2$.

Proof. Let $\{x_j\}$ be a sequence as in Lemma 1.1.2. Then by continuity, $\{\varphi_i(x_j)\}$ converges to $\varphi_i(y_i) \in Z$ for both $i = 1$ and $i = 2$, and our assertion follows. $\quad \square$

As such, we have

Corollary 1.1.5. *In the situation of* Proposition 1.1.4, *let* $z \in Z$. *Let* U *be a closed neighborhood of* z *in* Z. *Let* $Y_1(U)$ *denote the subset of* Y_1 *lying over* U, *and do likewise for* Y_2 *and* $\mathrm{LCM}(Y_1, Y_2)$. *Then*

$$\mathrm{LCM}(Y_1, Y_2)(U) = \mathrm{LCM}(Y_1(U), Y_2(U)). \quad \square$$

In particular, the fiber $\mathrm{LCM}(Y_1, Y_2)_z$ can be determined from data over any neighborhood of z.

Next, suppose there is a morphism, $\varphi : Y_1 \to Y_2$, and let Y_3 be any third element of $\mathfrak{PCp}(X)$. The induced mapping $\varphi \times 1 : Y_1 \times Y_3 \to Y_2 \times Y_3$ induces a morphism $\mathrm{LCM}(Y_1, Y_3) \to \mathrm{LCM}(Y_2, Y_3)$. It is important to recognize that there are no general

pullback properties in this situation, even in $\mathfrak{Cp}(X)$. Under the above conditions, one always has

(1.1.6) $$\mathrm{LCM}(Y_1, Y_3) \subseteq Y_1 \times_{Y_2} \mathrm{LCM}(Y_2, Y_3),$$

but equality can fail. In other words, the commutative diagram

$$
\begin{array}{ccc}
\mathrm{LCM}(Y_1, Y_3) & \longrightarrow & \mathrm{LCM}(Y_2, Y_3) \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
Y_1 & \xrightarrow{\ \varphi\ } & Y_2
\end{array}
$$

need not be Cartesian, even in $\mathfrak{Cp}(X)$.

(1.1.7) *Example.* A standard situation in which equality in (1.1.6) fails is given by taking a morphism $Y_1 \to Y_2$ in $\mathfrak{Cp}(X)$, and letting $Y_3 = Y_1$. Then $Y_1 \times_{Y_2} \mathrm{LCM}(Y_2, Y_3) = Y_1 \times_{Y_2} Y_1$, but $\mathrm{LCM}(Y_1, Y_3)$ is just Y_1. Indeed, equality holds in (1.1.6) if and only if $Y_1 \simeq Y_2$. (We add that it is easy to find examples in $\mathfrak{PCp}(X)$ of the preceding sort where equality in (1.1.6) holds without $Y_1 \simeq Y_2$. Indeed, take Y_1 and Y_2 as in (1.1.1.2), and $Y_3 = Y_2$.)

(1.1.8) *Definition.* We will say that LCM-*basechange holds for* Y_3 *with respect to* $Y_1 \to Y_2$ when we have equality in (1.1.6), i.e., $\mathrm{LCM}(Y_1, Y_3) = Y_1 \times_{Y_2} \mathrm{LCM}(Y_2, Y_3)$. When we want to emphasize the role of the boundaries (the notion is trivial on X) as in (1.1.3), we will say that LCMb-*basechange holds for* ∂Y_3 *with respect to* $\partial Y_1 \to \partial Y_2$.

When this is the case, the fiber of π_1 at $y \in Y_1$ is canonically homeomorphic to the fiber of π_2 at $\varphi(y)$.

(1.1.9) *Remark.* In terms of sequences, as in (1.1.2), LCM-basechange is equivalent to the following. Given a sequence in X with limit $y_2 \in Y_2$ and limit $y_3 \in Y_3$, then for any $y_1 \in Y_1$ that maps to $y_2 \in Y_2$, there is a sequence in X converging to $y_1 \in Y_1$ (so to $y_2 \in Y_2$) and to $y_3 \in Y_3$.

We have the following complement to Corollary 1.1.5:

Proposition 1.1.10. *In the situation of Proposition* 1.1.4,

$$\mathrm{LCM}(Y_1, Y_2)_z \subseteq (Y_1)_z \times (Y_2)_z,$$

with equality for all $z \in Z$ if and only if LCM-*basechange holds for Y_1 with respect to $Y_2 \to Z$.*

Proof. We note that by (1.1.6), $\mathrm{LCM}(Y_1, Y_2) \subseteq \mathrm{LCM}(Y_1, Z) \times_Z Y_2 = Y_1 \times_Z Y_2$, with equality if and only if the LCM-basechange assertion holds. The fiber of this over z is the same as that in the statement of the proposition. $\square$

Example (1.1.7) and the failure of surjectivity in morphisms in $\mathfrak{P}\mathfrak{C}\mathfrak{p}(X)$ suggest that we return to $\mathfrak{C}\mathfrak{p}(X)$ as setting for the rest of this Section. We next assert:

Proposition 1.1.11 (LCM-basechange in a tower)**.** *Let* $Y'' \to Y' \to Y$ *be morphisms of compactifications of* X, *and* Z *a fourth compactification of* X. *Then* LCM-*basechange holds for* Z *with respect to* $Y'' \to Y$ *if and only if* LCM-*basechange holds for* Z *with respect to both* $Y'' \to Y'$ *and* $Y' \to Y$.

Proof. From (1.1.6), we always have

$$(1.1.11.1) \quad \mathrm{LCM}(Y'', Z) \subseteq \mathrm{LCM}(Y', Z) \times_{Y'} Y''$$
$$\subseteq (\mathrm{LCM}(Y, Z) \times_Y Y') \times_{Y'} Y'' = \mathrm{LCM}(Y, Z) \times_Y Y''.$$

We see that there is equality of the ends if and only if we have equality at both inclusion symbols. This gives our assertion. $\square$

One can also talk about *two-sided* LCM-basechange. Let $Y' \to Y$ and $Z' \to Z$ be morphisms of compactifications of X. There is an embedding

$$(1.1.12) \qquad\qquad \mathrm{LCM}(Y', Z') \subseteq Y' \times_Y \mathrm{LCM}(Y, Z) \times_Z Z',$$

through which the embedding $\mathrm{LCM}(Y', Z') \subseteq Y' \times Z'$ factors.

Proposition 1.1.13. *Under the conditions of* (1.1.12), *equality holds in* (1.1.12) *if and only if* LCM-*basechange holds for* Z *and* Z' *with respect to* $Y' \to Y$ *and for* Y *and* Y' *with respect to* $Z' \to Z$.

Proof. This follows immediately from an elementary fact about fiber products: if A and B map to Z, and S is a proper subset of A, then $S \times_Z B$ is a proper subset of $A \times_Z B$. $\square$

(1.1.13.1) *Remark.* It is always the case that in the situation of (1.1.12),

$$\mathrm{LCM}(Y', Z') \simeq \mathrm{LCM}(\mathrm{LCM}(Y, Z'), \mathrm{LCM}(Y', Z)).$$

We give one instance of a simple and useful criterion for LCM-basechange. We return to the situation of (1.1.6).

Proposition 1.1.14. *Let $Y_1, Y_2, Y_3 \in \mathfrak{Cp}(X)$, with morphism $Y_1 \to Y_2$. Suppose that the compact Lie group H acts on the pairs (Y_1, X) and (Y_3, X), in such a way that $H \backslash \partial Y_1 \simeq \partial Y_2$. If H acts trivially on ∂Y_3, then LCM-basechange holds for Y_3 with respect to $Y_1 \to Y_2$.*

Proof. This is basically (1.1.9). We have that $\mathrm{LCM}(Y_1, Y_3) \to \mathrm{LCM}(Y_2, Y_3)$ is surjective. If a sequence $\{x_j\}$ in X satisfies $x_j \to y_2$ in Y_2 and $x_j \to y_3$ in Y_3, a subsequence converges to some $y_1 \in Y_1$, and then y_1 maps to y_2. Then for $h \in H$, $hx_j \to hy_1$ in Y_1; whereas in Y_3, $hx_j \to hy_3 = y_3$. $\square$

It is also possible for the opposite of LCM-basechange to occur. That happens whenever there is a morphism of compactifications $Y_3 \to Y_1$ (cf. (1.1.7)). More generally, one has:

Proposition 1.1.15. *In the situation of (1.1.6), $\mathrm{LCM}(Y_1, Y_3) \to \mathrm{LCM}(Y_2, Y_3)$ is an isomorphism if and only if the canonical mapping $\mathrm{LCM}(Y_2, Y_3) \to Y_2$ factors through Y_1.*

Proof. We are in the situation

$$
\begin{array}{ccc}
\mathrm{LCM}(Y_1, Y_3) & \longrightarrow & \mathrm{LCM}(Y_2, Y_3) \\
\downarrow & & \downarrow \\
Y_1 & \longrightarrow & Y_2
\end{array}
$$

When the top arrow is an isomorphism, its inverse can be used to define the factorization. Conversely, if we have a morphism $\mathrm{LCM}(Y_2, Y_3) \to Y_1$, use that with the canonical mapping $\mathrm{LCM}(Y_2, Y_3) \to Y_3$, to give a morphism $\mathrm{LCM}(Y_2, Y_3) \to \mathrm{LCM}(Y_1, Y_3)$. There is a canonical morphism in the opposite direction. This gives the isomorphism we were seeking. $\square$

Finally, we state a simple assertion that involves a second initial space:

Proposition 1.1.16. *Let X' be a space with a proper surjection $f : X' \to X$. Suppose that Y_1 and Y_2 are compactifications of X and Y_1' and Y_2' compactifications of X' for which f extends to mappings $f_1 : (Y_1', \partial Y_1') \to (Y_1, \partial Y_1)$ and $f_2 : (Y_2', \partial Y_2') \to (Y_2, \partial Y_2)$. Then the induced mapping $\mathrm{LCM}(Y_1', Y_2') \to \mathrm{LCM}(Y_1, Y_2)$ is surjective.*

Proof. If $\{x_j\}$ is a sequence converging in both Y_1 and Y_2, lift it to a sequence $\{x_j'\}$ in X'. The hypotheses implies that $\{x_j'\}$ has a subsequence that converges in both Y_1' and Y_2'. We are done by Lemma 1.1.2. $\square$

(1.2) Example: Two compactifications of a simplicial cone. The follow-
ing is an essential calculation, one that underlies [HZ1, 2.3], [HZ2, (1.5)], [GT,§7],
and what is to come in this article.

Let σ be a closed simplicial cone that spans a real vector space of dimension
d. When convenient, we will understand that the origin has been removed. Let
$\widehat{\sigma}$ denote the quotient of σ by cone dilations, a simplex that we can identify with
any cross-section of $\sigma \to \widehat{\sigma}$. Let $\{q_j\}$ be a set of d linear functionals that are
non-negative in σ and whose zero-loci define the codimension-one faces of σ. These
will be called *linear coordinates* on σ; they induce barycentric coordinates on $\widehat{\sigma}$. A
choice of cross-section enables us to write $\sigma \simeq \widehat{\sigma} \times (0, \infty)$ in the usual way. We
compactify σ to $\overline{\sigma}_1$ by taking $\overline{\sigma}_1 \simeq \widehat{\sigma} \times (0, \infty]$; thus, $\partial \overline{\sigma}_1 \simeq \widehat{\sigma}$.

A second compactification $\overline{\sigma}_2$ of σ is obtained by converting to *toroidal coordi-
nates*, by putting $t_j = \exp(-q_j)$. This sets up a homeomorphism $\sigma \simeq (0, 1]^d$, to
which we attach the boundary that yields $\overline{\sigma}_2 \simeq [0, 1]^d$. Then in toroidal coordinates

$$\partial \overline{\sigma}_2 \simeq \{x \in [0, 1]^d : t_j = 0 \text{ for some } j\} \simeq \bigcup \{\tau^\vee(\widehat{\sigma}) : \widehat{\tau} \text{ is a face of } \widehat{\sigma}\},$$

where $\tau^\vee(\widehat{\sigma})$ is the closed polyhedral face of $\overline{\sigma}_2$ dual to $\widehat{\tau}$. For τ a proper face of
σ, we define $\overline{\tau}_2$ to be the result of the preceding construction when one considers τ
as a subset of the linear space it spans. This allows us to regard $\overline{\tau}_2$ naturally as a
subset of $\overline{\sigma}_2$.

It is our present goal to determine $\mathrm{LCM}(\overline{\sigma}_1, \overline{\sigma}_2)$.

Proposition 1.2.1. *The complement of σ in* $\mathrm{LCM}(\overline{\sigma}_1, \overline{\sigma}_2) \subset \overline{\sigma}_1 \times \overline{\sigma}_2$ *is given by*

$$(1.2.1.1) \quad \mathrm{LCMb}(\widehat{\sigma}_1, \partial\overline{\sigma}_2) = \partial\mathrm{LCM}(\overline{\sigma}_1, \overline{\sigma}_2) = \bigcup \{\widehat{\tau} \times \tau^\vee(\widehat{\sigma}) : \widehat{\tau} \text{ is a face of } \widehat{\sigma}\}.$$

Proof. We make use of Lemma 1.1.1, together with the simple observation that q_j
goes to ∞ if and only if $t_j \to 0$. For each subset J of $\{1, 2, ..., d\}$, a face $\tau = \tau_J$ of σ
is defined by the equations $q_j = 0$ for all $j \in J$. Put $S = \sum_{1 \le j \le d} q_j$, and note that
for all j, the ratio $\widehat{q}_j := q_j/S$ takes values in $[0, 1]$. Suppose that $\{\mathbf{q}_k\}$ is a sequence
in the cone σ that converges to a boundary point $\mathbf{q}_\infty \in \widehat{\sigma} = \partial\overline{\sigma}_1$. Then $\mathbf{q}_\infty$ lies
in the interior $\widehat{\tau}_J^\circ$ of $\widehat{\tau}_J$ for a uniquely determined J. This implies for the sequence
$\{\mathbf{q}_k\}$ that $S \to \infty$, and thus the linear coordinates $q_j \to \infty$ for all $j \notin J$. In toroidal
coordinates, we have that $t_j \to 0$ for all $j \notin J$. As the face $\tau_J^\vee$ in $\partial\overline{\sigma}_2$ dual to $\widehat{\tau}_J$
is defined by the equations $t_j = 0$ for all $j \notin J$, we see that the right-hand side of
(1.2.1.1) contains the boundary of $\mathrm{LCM}(\overline{\sigma}_1, \overline{\sigma}_2)$.

To see that one gets every point of the latter space in the boundary, it suffices to exhibit a suitable family of curves in σ. For each J, consider the family of curves $\mathbf{q}_J(s)$ given in linear coordinates by

$$(1.2.1.2) \qquad\qquad q_j(s) = \begin{cases} s\, q_{j,0} & \text{if } j \notin J, \\ q_{j,0} & \text{if } j \in J, \end{cases}$$

with $q_{j,0} \in \mathbb{R}^{\geq 0}$ for all j. The limit in $\overline{\sigma}_1$, as $s \to \infty$, is the point of $\widehat{\tau}_J \times \{\infty\}$ with barycentric coordinates $\widehat{q}_j = 0$ if $j \in J$, and $\widehat{q}_j = \widehat{q}_{j,0}$ if $j \notin J$; in $\overline{\sigma}_2$, it is the point of $\tau_J^\vee$ with toroidal coordinates $t_j = \exp(-q_{j,0})$ if $j \in J$, and $t_j = 0$ if $j \notin J$. Thus, all of $\widehat{\tau} \times \tau^\vee(\widehat{\sigma})$ is reached. $\square$

We illustrate of the above when σ is of dimension $d = 2$. Let q_0 and q_1 be the non-negative linear functionals that define the edges of σ. We sketch four cone dilation orbits and a curve $t_1 = const$ (where q_1 is constant, so $\widehat{q}_1 \to 0$), which is of the sort given in (1.2.1.2), in both $\overline{\sigma}_1$ and $\overline{\sigma}_2$:

(1.2.2)

(1.2.3) *Remarks.* i) We can write (1.2.1.1) as

$$(1.2.3.1) \qquad\qquad \mathrm{LCMb}(\widehat{\sigma}_1, \partial\overline{\sigma}_2) = \bigcup \{\widehat{\tau}^\circ \times \tau^\vee(\widehat{\sigma}) : \widehat{\tau} \text{ is a face of } \widehat{\sigma}\},$$

where $\widehat{\tau}^\circ$ denotes the *interior* of $\widehat{\tau}$, consisting of those points in $\widehat{\tau}$ that do not belong to any proper face.

ii) It is not hard to see that the greatest common quotient of $\overline{\sigma}_1$ and $\overline{\sigma}_2$ is the one-point compactification of σ.

The picture that follows has been used in [HZ1: 2.3] and elsewhere, and will be used in the present work.

Construction 1.2.4. *There are quasi-canonical homeomorphisms of $\partial\overline{\sigma}_2$ and a (cubical) polyhedral decomposition $P(\widehat{\sigma})$ of $\widehat{\sigma}$ that subdivides further to the barycentric subdivision of $\widehat{\sigma}$.*

To see this, one notes that $\partial\overline{\sigma}_2$ is union of the d closed faces that pass through the origin of $[0,1]^d$. The barycentric subdivision of $\widehat{\sigma}$ consists of taking all simplices spanned by the *barycenters of* a chain of faces of $\widehat{\sigma}$:

$$(1.2.4.1) \qquad\qquad \tau_1 \subset \tau_2 \subset \ldots \subset \tau_m;$$

For the polyhedral complex, one takes only those chains where the codimension of each face in the next in (1.2.4.1) is equal to one. Removing the rest of the faces yields the polyhedral decomposition. The construction of a homeomorphism can be done recursively, by doing it first on all maximal faces, and then adjoining the origin of the cube and the barycenter of $\widehat{\sigma}$. If one subdivides $\partial\overline{\sigma}_2$ first, one can treat the t_j's as "linear" and make the homeomorphism simplicial and equivariant under permutation of the variables. The specification of the functionals q_j, which are unique up to positive scalars, determine this mapping uniquely.

The picture looks like a small piece of a Voronoi decomposition (see [N: p.98]). For a 2-simplex in the boundary when $d = 3$, the picture for $P(\widehat{\sigma})$ is:

(1.2.4.2)

(1.3) Discrete quotients and diagonality. We next consider the situation where there is a discrete group action, and the effect of that on LCM's.

Let D be a space on which a discrete group Γ acts. Let E be a partial compactification of D to which the action of Γ extends continuously. We suppose that the action on E (so also its restriction to D) is *separated and discontinuous*, by which we mean that the following two conditions hold:

i) if $y \in E$ is not a Γ-translate of x, then there are neighborhoods U_x of x and U_y of y with $U_y \cap (\Gamma \cdot U_x) = \emptyset$.

ii) if $x \in E$, there is a neighborhood U_x of x in E such that if $\gamma \in \Gamma$ and $\gamma \cdot x \in U_x$ then $\gamma \cdot x = x$.

Suppose we have two such E—call them E_1 and E_2. Then $\Gamma \times \Gamma$ acts on $E_1 \times E_2$. The diagonal $\Delta(\Gamma)$, which we identify with Γ, is easily seen to preserve the subset $\mathrm{LCM}(E_1, E_2)$. The projections of $\mathrm{LCM}(E_1, E_2)$ onto E_1 and E_2 are equivariant for the Γ-actions. Moreover, via either projection, one can easily deduce that the action on $\mathrm{LCM}(E_1, E_2)$ is separated and discontinuous. We want a little more.

(1.3.1) *Definition.* i) We say that the actions of Γ on E_1 and E_2 are *diagonal* when the following holds: if $(e_1, e_2) \in \mathrm{LCM}(E_1, E_2)$ and $(\gamma_1 \cdot e_1, \gamma_2 \cdot e_2) \in \mathrm{LCM}(E_1, E_2)$ for some $\gamma_1, \gamma_2 \in \Gamma$, then there exists $\delta \in \Delta(\Gamma)$ with $\delta \cdot e_1 = \gamma_1 \cdot e_1$ and $\delta \cdot e_2 = \gamma_2 e_2$; equivalently, if $(e_1, e_2) \in \mathrm{LCM}(E_1, E_2)$ and $(\gamma \cdot e_1, e_2) \in \mathrm{LCM}(E_1, E_2)$, there exists $\delta \in \Delta(\Gamma)$ with $e_1 = (\delta\gamma) \cdot e_1$ and $e_2 = \delta \cdot e_2$, i.e., $(\gamma \cdot e_1, e_2) = \delta \cdot (e_1, e_2)$.

ii) We say that the actions of Γ on E_1 and E_2 are *strongly diagonal* when the

following holds: if $(e_1, e_2) \in \mathrm{LCM}(E_1, E_2)$ and $(\gamma \cdot e_1, e_2) \in \mathrm{LCM}(E_1, E_2)$, then $\gamma \cdot e_1 = e_1$ or $\gamma \cdot e_2 = e_2$.

Strongly diagonal actions are diagonal. We also say that *diagonality holds* under the conditions of (1.3.1). We prefer strong diagonality in the sequel. It is easy to see that diagonality is not affected by switching E_1 and E_2. We will give a rather simple, and quite pertinent, example of non-diagonal actions in (1.3.6) below.

(1.3.2) *Remark.* It is easy to see that if there is a Γ-equivariant morphism $E_1 \to E_2$ in $\mathfrak{PCp}(D)$, then $\mathrm{LCM}(E_1, E_2) \simeq E_1$ is diagonal.

As the non-trivial issues lie at the boundaries of E_1 and E_2, we also speak of *boundary diagonality*, in the spirit of the notion of $\mathrm{LCMb}(E_1, E_2)$ from (1.1.3). A little surprisingly, the useful properties of diagonality that we have found are set-theoretical, i.e., not topological in nature. We therefore make the following generalization of (1.3.1):

(1.3.3) *Definition.* Let E and E' be sets given with actions by a group Γ, so that there is a natural action of $\Gamma \times \Gamma$-action on $E \times E'$. Let B be a $\Delta(\Gamma)$-invariant subset of $E \times E'$. We say that B is *diagonal* if (e, e'), $(\gamma \cdot e, e') \in B$ implies that $\gamma \cdot e = e$ or $\gamma \cdot e' = e'$. In other words, *on B, $\Gamma \times \Gamma$-equivalence reduces to $\Delta(\Gamma)$-equivalence.*

The following is immediate:

Lemma 1.3.4. *i) Let $B_1 \subset B_2$ be $\Delta(\Gamma)$-invariant subsets of $E_1 \times E_2$, with B_2 diagonal. Then B_1 is also diagonal.*

ii) Let $\Gamma' \subset \Gamma$. If B is diagonal for Γ, it is diagonal for Γ'. □

In the original setting of E, $E' \in \mathfrak{PCp}(D)$, we understand that $B = \mathrm{LCM}(E, E')$ in (1.3.3) unless stated otherwise.

(1.3.5) *Remark.* We comment on diagonality in an important setting related to LCM-basechange. Let Γ be a group acting on sets E_1, E_2 and E_3. Assume that a Γ-equivariant mapping $E_3 \to E_1$ is specified. *We now take $E_1 \times E_2$ to have the diagonal Γ-action.* Consider the mappings

(1.3.5.1) $\qquad E_3 \times_{E_1} (E_1 \times E_2) \subset E_3 \times (E_1 \times E_2) \to (E_3 \times E_2) \to (E_1 \times E_2).$

The projection $E_3 \times (E_1 \times E_2) \to E_3 \times E_2$ is $(\Gamma \times \Gamma)$-equivariant; however, the subset $E_3 \times_{E_1} (E_1 \times E_2)$ of the domain is only $\Delta(\Gamma)$-invariant. It is not true, for instance, that B diagonal in $E_1 \times E_2$ implies $E_3 \times_{E_1} B$ diagonal in $E_3 \times E_2$.

We give next an example of a pair of actions that are non-diagonal in the sense of (1.3.1), one that fits well with the considerations of (3.2):

(1.3.6) *Example.* Let E_1 be the real line, partitioned at the integer points. This gives a one-dimensional simplicial complex with 1-simplices $[n, n+1]$ for all $n \in \mathbb{Z}$. The group $\Gamma = \mathbb{Z}$ acts on E_1 in the usual way, by translation. Let E_2 be the dual cell complex, which is $\mathbb{R}$ with 1-cells $[n - \frac{1}{2}, n + \frac{1}{2}]$, with the same Γ-action. Let

$$B = \bigcup \{\tau \times \tau^\vee : \tau = [n, n+1], \, n \in \mathbb{Z}\}$$

(cf. (1.2.3.1)). Then the points $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ are in B, though 1 is a Γ-translate of 0, and Γ has no fixed points in either E_1 or E_2. Note, however, that B *is* diagonal for the action of, say, $2\mathbb{Z}$.

The next assertion covers a rather common situation.

Proposition 1.3.7 (Diagonality and products)**.** *Let $D = D_1 \times D_2$, the product of two spaces. Suppose that the discrete group $\Gamma = \Gamma_1 \rtimes \Gamma_2$ (semi-direct product, Γ_1 normal in Γ) acts factorwise on D. Let $E = E_1 \times E_2$ and $E' = E'_1 \times E'_2$, for Γ_1-equivariant $E_1, E'_1 \in \mathfrak{PCp}(D_1)$, and Γ_2-equivariant $E_2, E'_2 \in \mathfrak{PCp}(D_2)$. Assume that diagonality holds for the actions of Γ_1 on E_1 and E'_1, and for the actions of Γ_2 on E_2 and E'_2. Then the actions of Γ on E and E' are diagonal, and $\mathrm{LCM}(\Gamma \backslash E, \Gamma \backslash E') \simeq \Gamma \backslash \mathrm{LCM}(E, E')$ is a $\Gamma_2 \backslash \mathrm{LCM}(E_2, E'_2)$-fibration over $\Gamma_1 \backslash \mathrm{LCM}(E_1, E'_1)$.*

Proof. We begin by taking the Γ_2-quotients. That gives at once (one can use Lemma 1.1.2):

$$\mathrm{LCM}(E_1 \times (\Gamma_2 \backslash E_2), E'_1 \times (\Gamma_2 \backslash E'_2)) \simeq \mathrm{LCM}(E_1, E'_1) \times \mathrm{LCM}((\Gamma_2 \backslash E_2), (\Gamma_2 \backslash E'_2))$$
$$\simeq \Gamma_2 \backslash \mathrm{LCM}(E, E').$$

Taking the quotient by $\Gamma / \Gamma_2 \simeq \Gamma_1$ (isomorphism of groups), we obtain the result. $\square$

The following extension of Proposition 1.1.16 combines the above notions, and is tailored to our later needs in §3:

Proposition 1.3.8. *Let D be a space with a separated discontinuous action of a discrete group Γ, and let $X = \Gamma \backslash D$. Suppose that E_1 and E_2 are partial compactifications of D, yielding spaces for which the Γ actions are diagonal. Assume that there is a subset S of D that maps onto X, and whose respective closures S_1, S_2 in*

E_1 and E_2 are compact; thus $Y_1 = \Gamma \backslash E_1$ and $Y_2 = \Gamma \backslash E_2$ are compactifications of X. Then the canonical mapping

$$\Gamma \backslash \mathrm{LCM}(E_1, E_2) \to \mathrm{LCM}(Y_1, Y_2)$$

(where Γ acts on $\mathrm{LCM}(E_1, E_2)$ as $\Delta(\Gamma)$) is a homeomorphism.

Proof. Let $\{x_j\}$ be a sequence in X that converges in both Y_1 and Y_2. The existence of S allows the lifting of $\{x_j\}$ to S, such that a subsequence converges in S_1 and S_2, a fortiori in E_1 and E_2. Thus $\mathrm{LCM}(E_1, E_2) \to \mathrm{LCM}(Y_1, Y_2)$ is surjective. We wish to show that the fibers are $\Delta(\Gamma)$-equivalence classes. So, suppose that (e_1, e_2) and (e_1', e_2') have the same image in $\mathrm{LCM}(Y_1, Y_2)$. Then $e_1' = \gamma_1 \cdot e_1$ and $e_2' = \gamma_2 \cdot e_2$ for $\gamma_1, \gamma_2 \in \Gamma$. By diagonality, we can take $\gamma_1 = \gamma_2$, and we are done. $\square$

2. Locally-symmetric varieties and their compactifications

In this section we present key elements of the structure of the relevant compactifications of locally symmetric varieties. (A list of notational changes from [Z4] is provided at the end of the introduction.)

(2.1) Boundary components and the Baily-Borel compactification. Let G be a semi-simple algebraic group defined over the rational number field $\mathbb{Q}$. For P a parabolic subgroup of $G/\mathbb{Q}$ ($\mathbb{Q}$-*parabolic subgroup*; we will henceforth suppress the "$\mathbb{Q}$"), let W_P denote the unipotent radical of P. The *Levi quotient* of P is $P^{\mathrm{red}} = P/W_P$, and the projection $P \to P^{\mathrm{red}}$ is split by a choice of Levi subgroup $L_P \subset P$.

Let D be the symmetric space of non-compact type associated to $G(\mathbb{R})$. For each parabolic subgroup P, its real points act transitively on D. Unless specified otherwise, *we assume throughout that $G(\mathbb{R})$ is of Hermitian type*, i.e., that D has a $G(\mathbb{R})$-invariant complex structure.

We further assume that G is (almost) simple over $\mathbb{Q}$. Then for each maximal parabolic P, L_P is an almost-direct product of two groups, commonly denoted $G_{h,P}$ and $G_{\ell,P}$ after [AMRT: p.209]. The rational boundary component D_P of D that is normalized by P is biholomorphic to the Hermitian symmetric space associated to $G_{h,P}$. On the other hand, the group $G_{\ell,P}$ (rarely of Hermitian type) acts trivially on D_P, as does W_P. The set

(2.1.1) $$D^* = D \sqcup \bigsqcup \{D_P : P \text{ is maximal parabolic}\}$$

has a canonical $G(\mathbb{Q})$-action. It is given the $G(\mathbb{Q})$-equivariant *Satake topology* [S,§2] (see [Z3:(3.9)]).

The $\mathbb{Q}$-root system of G is of classification type BC or C [BB:2.9], whose Dynkin diagram is a linear graph with distinguished end. The set Δ of simple roots is thus totally ordered by specifying the root at the distinguished end to be the minimal element. In a standard lattice of parabolic subgroups, the specification of a maximal parabolic subgroup P is equivalent to selecting one simple root $\beta_P \in \Delta$; the correspondence is: the root space for a root β is contained in W_P if and only if β_P occurs with positive coefficient in the expansion of β. One says that $Q \prec P$ whenever $\beta_Q \prec \beta_P$. Moreover, the Dynkin diagram for $G_{h,P}$ is the segment $\Delta_{h,P}$ of type (B)C below β_P (in the total ordering), and that of $G_{\ell,P}$ is the segment $\Delta_{\ell,P}$ of type A above β_P. Thus, one can assert:

Proposition 2.1.2. *When G is simple over $\mathbb{Q}$, the following are equivalent for maximal parabolic subgroups: (i) $Q \prec P$, (ii) D_Q is a boundary component of D_P, (iii) $G_{h,Q} \subset G_{h,P}$, (iv) $G_{\ell,Q} \supset G_{\ell,P}$.* $\quad\square$

Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. We assume throughout that Γ is neat, i.e., that for every algebraic subquotient group H of G, the induced subquotient of Γ, denoted $\Gamma(H)$ (instead of the "Γ_H" that appears frequently in the literature), is torsion-free. The latter for $H = G$ already gives that the quotient space $X = \Gamma\backslash D$, called a *locally-symmetric* (or *arithmetic*) *variety*, is non-singular. We are interested in some of its compactifications.

The *Baily-Borel Satake compactification* of X is

(2.1.3) $$X^* = \Gamma\backslash D^*,$$

homeomorphic in the sense of (1.1) to the Satake compactification of X that is parametrized by the root at the distinguished end (see [Z3:(3.9)]). It is shown in [BB] that X^* is a projective algebraic variety (of which X is the regular locus), whence the word "variety" in the name for X. The decomposition (2.1.1) induces a stratification of (2.1.3) as follows. For P as above, a boundary stratum

(2.1.4) $$M_P = \Gamma(P)\backslash D_P$$

of X^* is induced by the inclusion $D_P \subset D^*$ in (2.1.1). For $Q \prec P$, M_Q lies in the closure of M_P in X^*, the latter giving the Baily-Borel compactification M_P^* of M_P. The quotient by Γ identifies the boundary components of Γ-conjugate

P's, so boundary strata in X^* are parametrized by Γ-*conjugacy classes* of maximal parabolic subgroups, and these are finite in number.

(2.2) Toroidal compactifications and their quotients. The toroidal compactifications X^{tor} of [AMRT] are predicated on the notion that they will map to the Baily-Borel compactification X^*. As the construction is rather complicated, we give here only a brief description, referring to the literature for details.

For P maximal parabolic, the center U_P of W_P plays a key role in the construction; the quotient $V_P = W_P/U_P$ is commutative and even-dimensional, as follows from the root structure. It is elementary but fundamental that $U_Q \supset U_P$ whenever $Q \prec P$ (there is no corresponding assertion for the whole unipotent radical).

For each P, there is a tower of quotient mappings of groups

$$(2.2.1) \qquad P' = G_{h,P}(\mathbb{R})W_P(\mathbb{R})U_P(\mathbb{C}) \to G_{h,P}(\mathbb{R})V_P(\mathbb{R}) \to G_{h,P}(\mathbb{R}).$$

These act homogeneously on a tower of spaces $\check{D}(P) \to D_P^A \to D_P$, with common isotropy subgroup $K_{h,P}$, which is maximal compact in $G_{h,P}(\mathbb{R})$. From (2.2.1) and the Cayley transform associated to P comes the Siegel domain picture of D relative to D_P (see [AMRT:III,§4], and also our (4.1)). Then, dividing out the respective actions of $\Gamma(G_{h,P}W_P)$, $\Gamma(G_{h,P}V_P)$, and $\Gamma(G_{h,P})$, one gets the basic tower (essentially of mixed Shimura varieties; see [HZ1:1.6]):

$$(2.2.2) \qquad\qquad\qquad M_P' \xrightarrow{\pi_2} \mathcal{A}_P \xrightarrow{\pi_1} M_P.$$

(The use of "prime" in M' and P' differs from convention that begins in (2.3).)

We underscore the absence of $G_{\ell,P}$ in (2.2.1). (It disappears through inclusion in the isotropy subgroup for a suitable basepoint at infinity.) The action of $\Gamma(G_{\ell,P})$ on (2.2.2) is induced by conjugation on the groups in (2.2.1). It is trivial on M_P and free on M_P'. As π_1 is a fibration by abelian varieties, in particular proper, the $\Gamma(G_{\ell,P})$-orbits in $\mathcal{A}_P$ are rather ugly, except in the case that $\mathcal{A}_P = M_P$ (i.e., when W_P is commutative). It is an essential observation that π_2 is a principal torus bundle with fiber

$$(2.2.3) \qquad\qquad\qquad T_P = \Gamma(U_P)\backslash U_P(\mathbb{C}).$$

Through (2.2.3), one uses torus embeddings to attach a substantial boundary to M_P' for each P, in a fashion that is both equivariant for the action of $\Gamma(G_{\ell,P})$ and

compatible with order relations and conjugacy among maximal parabolic subgroups [AMRT: p.117]. One then uses reduction theory to do the same for $X = \Gamma \backslash D$, and this produces the desired compactification of X [AMRT: III,§§5,6].

We want to specify, and comment on, some of the details. The space X^{tor} depends on a collection Σ of simplicial cone complexes (*fans*), $\Sigma_P \subset U_P(\mathbb{R})$ (as P varies), such that for $^\gamma P = \gamma P \gamma^{-1}$ (with $\gamma \in \Gamma$), $\Sigma_{\gamma P} = \gamma(\Sigma_P)\gamma^{-1}$; also, whenever $Q \prec P$, Σ_Q contains Σ_P as a closed boundary stratum (in a precise sense; see below), and $\Gamma(G_{\ell,P})$ acts separated and discontinuously on Σ_P, compatibly with the order relation. We always assume that Σ_P is sufficiently fine so as to have *full boundary*, in the sense of [HZ1:2.2.6], to avert combinatorial anomalies.

The use of the term *cone complex* in the above means that a fan Υ is subject to the following two axioms:

A1. *If $\sigma \in \Upsilon$, and τ is a face of σ, then $\tau \in \Upsilon$.*

A2. *If $\sigma, \sigma' \in \Upsilon$, and $\sigma \cap \sigma' \neq \{0\}$, then $\sigma \cap \sigma'$ is a face of both σ and σ'.*

Any fan Υ in $U_P(\mathbb{R})$ determines in a standard way a torus embedding $T_P \subset T_{P,\Upsilon}$, on which T_P acts (see [HZ1: 1.3] and references cited therein). Briefly, this goes as follows; we present it for an arbitrary torus T. Each cone σ determines an affine torus embedding $T = T_{\{0\}} \subset T_\sigma$ (T-equivariant partial compactification), such that $T_\tau \subset T_\sigma$ whenever τ is a face of σ. The axioms allow for gluing, viz., if $\tau = \sigma \cap \sigma'$,

$$(2.2.4) \qquad\qquad T_{\{\sigma,\sigma'\}} = T_\sigma \cup_{T_\tau} T'_\sigma.$$

One takes $T_\Upsilon = \bigcup \{T_\sigma : \sigma \in \Upsilon\}$, with identifications as given by (2.2.4).

The notation in (2.2.4) contains the incidental statement that $T_{\{\sigma,\sigma'\}}$ is determined only by σ and σ' (and their intersection). Moreover, nothing new is obtained when σ' is a face of σ; σ and the fan it generates (i.e., the one consisting of σ and all of its faces) produce the same torus embedding. Thus, axiom A1, which provides for gluing, can be relaxed. We say that a collection of cones Υ is a *loose fan* if it satisfies the following weakened version of the axioms of a fan:

A2′. *If $\sigma, \sigma' \in \Upsilon$, then $\sigma \cap \sigma'$ is in Υ (so is a face of both σ and σ').*

A loose fan Υ generates a fan $\overline{\Upsilon}$, given as the set of all elements of Υ and their faces. It also determines a torus embedding T_Υ, by the same procedure that is specified for fans. It is clear that $T_\Upsilon = T_{\overline{\Upsilon}}$.

It is convenient to remove the origin and contract out the cone dilations in a fan, yielding from Υ a simplicial complex $\widehat{\Upsilon}$; likewise, we will take off our hats to

get the fans, i.e., pass routinely from $\widehat{\Upsilon}$ to Υ when $\widehat{\Upsilon}$ is mentioned first. Moreover, we standardize notation by using σ, τ, etc. for simplicial cones, and $\widehat{\sigma}$, $\widehat{\tau}$, etc. for simplices. Each cone $\sigma \in \Upsilon$ corresponds to a T-orbit, which we denote $T(\sigma)$, with $\operatorname{codim}_{\mathbb{C}} T(\sigma) = \dim_{\mathbb{R}} \sigma$, and then

$$(2.2.4.1) \qquad T_\sigma = \bigsqcup \{T(\tau) : \tau \text{ is a face of } \sigma\};$$

here we allow the improper face ($\tau = \sigma$). We note that $\widehat{\Upsilon}$ is isomorphic to the nerve of the set of closed (i.e., closures of) T-orbits in T_Υ, via the correspondence $\widehat{\tau} \mapsto \tau^\vee$ (cf. (1.2); see also (3.2.4.1) below).

With $T = T_P$ and $\Upsilon = \Sigma_P$, the above construction determines a partial compactification $M'_{P,\Upsilon}$ of M'_P, with

$$(2.2.5.1) \qquad M'_{P,\Upsilon} = M'_P \times^{T_P} T_{P,\Upsilon} \to \mathcal{A}_P \to M_P.$$

The mapping onto $\mathcal{A}_P$ extends π_1 in (2.2.2); the restriction of (2.2.5.1) to the boundary is

$$(2.2.5.2) \qquad \partial M'_{P,\Upsilon} =: \widetilde{Z}_{P,\Upsilon} = M'_P \times^{T_P} \partial T_{P,\Upsilon} \to \mathcal{A}_P \to M_P.$$

We refer to the preceding as the *toroidal construction from* Υ (and T_P and M'_P). Then $\widehat{\Upsilon}$ is isomorphic to the nerve of the set of all irreducible components of $\widetilde{Z}_{P,\Upsilon}$, as it is already true for T_Υ. Because we are taking only simplicial fans $\partial T_{P,\Upsilon}$, hence also $\widetilde{Z}_{P,\Upsilon}$, is a union of smooth divisors with normal crossings.

Given a subset Σ of the fan Υ, we define a subset $T(\Sigma)$ of T_Υ by:

$$(2.2.6) \qquad T(\Sigma) = \bigsqcup \{T(\sigma) : \sigma \in \Sigma\};$$

note that with this notation, $T(\Upsilon) = T_\Upsilon$. The space $T(\Sigma)$ is invariant under T, and it therefore determines a subset of $\widetilde{Z}_{P,\Upsilon}$:

$$(2.2.7) \qquad \widetilde{Z}(\Sigma) = M'_P \times^T \partial T(\Sigma).$$

We allow ourselves to write $T(\widehat{\Sigma})$ and $\widetilde{Z}(\widehat{\Sigma})$, and the stratification describe instead of $T(\Sigma)$ and $\widetilde{Z}(\Sigma)$ resp.

(2.2.8) *Remark.* There is a familiar and simple characterization of when $\partial T(\widehat{\Sigma})$ is closed in $\partial T(\widehat{\Upsilon})$. It involves the *star* of $\widehat{\Sigma}$ in $\widehat{\Upsilon}$, given as

$$\operatorname{Star}(\widehat{\Sigma}) = \{\tau \in \widehat{\Upsilon} : \text{ a vertex of } \widehat{\tau} \text{ is contained in } \widehat{\Sigma}\}.$$

Then the closure of $\partial T(\widehat{\Sigma})$ in $\partial T(\widehat{\Upsilon})$ is $\partial T(\mathrm{Star}(\widehat{\Sigma}))$; thus, $\partial T(\widehat{\Sigma})$ is closed if and only if the inclusion $\widehat{\Sigma} \subseteq \mathrm{Star}\,(\widehat{\Sigma})$ is an equality. In particular, when Σ is a proper subfan of Υ, $\partial T(\widehat{\Sigma})$ is never closed. (See also (2.2.10) below.)

The interior of Σ_P is, in fact, $\Gamma(G_{\ell,P})$-equivariantly homeomorphic to the homogeneous cone C_P occurring in the Siegel domain mentioned in conjunction with (2.2.2), an orbit of the adjoint representation of $G_{\ell,P}$ on $U_P(\mathbb{R})$, which is how it enters the construction. Let $D_{\ell,P}$ be the space of type $S - \mathbb{Q}$ (in the sense of [BS: 2.3]) for $\widehat{G}_{\ell,P} = G_{\ell,P}/A_P$, with A_P as in (2.3) below ($D_{\ell,P}$ need not be a symmetric space, as $\widehat{G}_{\ell,P}$ may contain central anisotropic tori). Put $X_{\ell,P} = \Gamma(G_{\ell,P})\backslash D_{\ell,P}$ and $\widehat{\Sigma}'_P = \Gamma(G_{\ell,P})\backslash \widehat{\Sigma}_P$. The complex $\widehat{\Sigma}'_P$ is actually a stratified triangulation of a Satake compactification of $X_{\ell,P}$ (in the sense of [Z3: §3] and [HZ2: (2.1)]). As such, $\widehat{\Sigma}'_P$ is a non-Hermitian analogue of M_P^*, which is itself a Satake compactification of M_P (see [Z3:(3.9)]).

We mention two good choices for the fan Υ: Σ_P and its subfan Σ_P^c; between them lies the set Σ_P°. All three are closed under the action of $\Gamma(G_{\ell,P})$ on Σ_P. We give the definition of the latter two: Σ_P^c is the fan spanned by the interior edges of Σ_P, and Σ_P° consists of those cones in Σ_P that contain at least one edge from Σ_P^c. Thus, Σ_P° is usually not even a loose fan. In terms of simplices,

$$(2.2.9) \qquad \widehat{\Sigma}_P^\circ = \widehat{\Sigma}_P - \bigcup\{\widehat{\Sigma}_Q : Q \succ P\}, \quad \text{so} \quad \widehat{\Sigma}_P = \bigsqcup\{\widehat{\Sigma}_Q^\circ : Q \succeq P\}.$$

(2.2.10) *Remark.* We illustrate $\widehat{\Sigma}_P^c$ and $\widehat{\Sigma}_P^\circ$ by describing them in the case where $\widehat{\Sigma}_P$ is replaced by the fan generated by a one-simplex, which consists of two vertices $[0]$ and $[1]$ spanning the simplex $[0,1]$. We declare that $[1]$ is interior and $[0]$ is boundary (if this were contained in $\widehat{\Sigma}_P$, then there would be $Q \succ P$ with $[0] \in \widehat{\Sigma}_Q$). Then:

$$\widehat{\Sigma}_P^c = \{[1]\}, \quad \widehat{\Sigma}_P^\circ = \{[1], [0,1]\}, \quad \text{and} \quad \widehat{\Sigma}_P = \{[0], [1], [0,1]\}.$$

The boundary of X^{tor} stratifies naturally in terms of Σ. We admit with some apology that the notion of *boundary stratum* changed during the progression from [HZ1] to [HZ2]. Also, the term "stratum" is being used loosely, in that our strata need not be smooth.

The *Baily-Borel-type P-stratum* in X^{tor}, denoted $^<Z_P$ (cf. [HZ1: 1.5]), is the $\Gamma(G_{\ell,P})$-quotient of $^<\widetilde{Z}_P := \widetilde{Z}(\Sigma_P^\circ)$; this in fact equals the closure of $\widetilde{Z}(\Sigma_P^c)$ in $\widetilde{Z}(\Sigma_P)$. It follows directly from the construction that

$$\widetilde{Z}(\Sigma_P) = \bigsqcup\{^<\widetilde{Z}_Q : Q \succeq P\}$$

(for such Q, Σ_Q is a boundary component of Σ_P). In fact, $^<\widetilde{Z}_Q$ is produced in the toroidal construction from $\widehat{\Sigma}_Q$ (and T_Q and M'_Q), and is partially compactified in the toroidal construction from $\widehat{\Sigma}_P \supset \widehat{\Sigma}_Q$. We let Z_P denote the closure of $^<Z_P$ in X^{tor}, and refer to it as the *closed P-stratum*. The complement of $^<Z_P$ in Z_P is the (disjoint) union of all $^<Z_Q \cap Z_P$ with $Q \prec P$—now with $\widehat{\Sigma}_P$ a boundary component of $\widehat{\Sigma}_Q$. Indeed, the combinatorial data for constructing Z_P is contained in the union of the *open* regular neighborhoods of $\widehat{\Sigma}_P^c$ in $\widehat{\Sigma}_Q$, as $Q \preceq P$ varies (see [HZ2:(2.5.1,i)]); when $Q = P$, this neighborhood coincides with $\widehat{\Sigma}_P^\circ$. Under the natural mapping $\pi : X^{\mathrm{tor}} \to X^*$ [AMRT:p.254], $\pi^{-1}(M_P) = {}^<Z_P$, and then $\partial X^{\mathrm{tor}} = \bigsqcup_P {}^<Z_P$, with the union taken over Γ-conjugacy classes.

Next, let R be an arbitrary parabolic subgroup (possibly maximal) of G. We want to specify the (cubical) *R-stratum* $\overset{\circ}{Z}_R$ in X^{tor}. One can write R uniquely as an intersection of maximal parabolic subgroups in a standard lattice:

$$(2.2.11) \qquad R = \bigcap \{Q : Q \in \mathcal{S}\}, \quad \text{where}$$

$$\mathcal{S} = \mathcal{S}_R = \{Q \text{ maximal} : Q \supseteq R\} \leftrightarrow \{\beta \in \Delta : \beta = \beta_Q \text{ for some maximal } Q \supseteq R\}.$$

(Note that this usage of subsets of Δ is complementary to the usual one for indexing parabolic subgroups; a minimal parabolic subgroup corresponds to Δ here, not $\emptyset$.) Let P be the smallest element in $\mathcal{S}$. As in [HZ2:(2.2)], we then say that R is *subordinate to P*, and write $P = \Pi(R)$. The R that are subordinate to a given P are in canonical one-to-one correspondence with the standard parabolic subgroups $G_{\ell,R}$ of $G_{\ell,P}$ relative to $\Delta_{\ell,P}$, including the improper one, by taking

$$(2.2.12) \qquad G_{\ell,R} = R \cap G_{\ell,P}, \quad \text{so then} \quad R = G_{\ell,R} \cdot G_{h,P} \cdot W_P.$$

If there were a good way to remove the Γ-quotient from X^{tor} to yield a space we might call $\widetilde{X}^{\mathrm{tor}}$, the closed strata Z_P of X^{tor} would be induced from their inverse images $\widetilde{Z}_P$ in $\widetilde{X}^{\mathrm{tor}}$. The stratification would be *cubical* (or *corner-like*), in that the closed strata $\{\widetilde{Z}_R\}$ of $\widetilde{X}^{\mathrm{tor}}$ would satisfy

$$(2.2.13) \qquad \widetilde{Z}_R = \bigcap\{\widetilde{Z}_Q : Q \in \mathcal{S}\}$$

(compare (2.3.3) below), so the strata would be deducible from (2.2.13). However, the preceding is an oversimplification.

Here is one way to proceed. Define for each R with $\Pi(R) = P$ a subset of $\widehat{\Sigma}_P$:

$$(2.2.14) \quad \widehat{\Sigma}_R^c := \{\widehat{\tau} \in \widehat{\Sigma}_P \,|\, \widehat{\tau} \text{ has all vertices in } \bigcup_{Q \in \mathcal{S}} \widehat{\Sigma}_Q^c \text{ and a vertex in each } \widehat{\Sigma}_Q^c\}.$$

This agrees with what we gave earlier when R is maximal. As R varies (subordinate to P), the $\widehat{\Sigma}_R^c$'s are disjoint.

(2.2.15) *Definition.* Let $\check{Z}_R = \widetilde{Z}(\widehat{\Sigma}_R^c)$. The *R-stratum* of ∂X^{tor} is $\overset{\circ}{Z}_R \simeq \Gamma(G_{\ell,R})\backslash\check{Z}_R$.

Note that replacing R by any Γ-conjugate of R gives the same stratum; they are otherwise disjoint. We denote by Z_R the closure of $\overset{\circ}{Z}_R$ in ∂X^{tor} (the closed R-stratum). Because Z_R contains points from more than one standard lattice, it is somewhat inconvenient to express it in terms of torus orbits.

We note that one has for the Baily-Borel-type P-stratum

(2.2.16.1) $$ {}^<Z_P = \bigsqcup \{ \overset{\circ}{Z}_{R'} : \Pi(R') = P \}. $$

This suggests that one might also put, for general R,

(2.2.16.2) $$ {}^<Z_R = \bigsqcup \{ \overset{\circ}{Z}_{R'} : R' \subseteq R, \Pi(R') = P \}, $$

which gives the portion of Z_R that is created in the toroidal construction from $\widehat{\Sigma}_P$.

(2.2.17) *Remark.* Z_R is generally only a connected component of $\bigcap \{Z_Q : Q \in \mathcal{S}\}$. Because of the Γ-quotient, this intersection usually has more than one connected component, only one of which one wants to associate with R. The others correspond to certain $G(\mathbb{Q})$-conjugates of R (see [HZ2:(3.5, App.)]).

In [HZ2:(1.4, (d))], we introduced a canonical (topological) quotient of X^{tor}, called its *real boundary quotient*. Specifically, one collapses ${}^<Z_P$ (2.2.16.1) of X^{tor} to ${}^<Z_P/T_P^c$, which inherits the fibrations (2.2.5.2) over $\mathcal{A}_P$ and M_P; here, T_P^c denotes the compact factor of T_P, viz., $\Gamma(U_P)\backslash U_P(\mathbb{R})$. Because its construction is parallel to that of the excentric Borel-Serre compactification [HZ2:(1.4, (b))] (see (2.3) below), we rename it an *excentric toroidal compactification* of X, and denote it $X^{\text{tor,exc}}$. We get a tower of compactifications (for each Σ):

(2.2.18) $$ X^{\text{tor}} \to X^{\text{tor,exc}} \to X^*. $$

Likewise, we write ${}^<Z_P^{\text{exc}}$ for ${}^<Z_P/T_P^c$ and $\overset{\circ}{Z}_R^{\text{exc}}$ for the R-stratum of $X^{\text{tor,exc}}$, viz. $\overset{\circ}{Z}_R/T_P^c$; define $\check{Z}_R^{\text{exc}}$ analogously as $\check{Z}_R/T_P^c$. We can see that the topological structure of $\check{Z}_R^{\text{exc}}$ is rather simple. First, we make a basic observation:

Proposition 2.2.19 [HZ1: 2.1]**.** *Let* $T \simeq (\mathbb{C}^*)^n$ *be a torus, so that* $T/T^c \simeq (\mathbb{R}^+)^n$. *Given any simplicial fan* Υ *for* T, *let* T_Υ *denote the corresponding torus embedding. Then* T_Υ/T^c *is a contractible n-dimensional manifold-with-corners.* $\quad\square$

From this, we see immediately that

Corollary 2.2.20. *The natural mapping $\check{Z}_R^{\mathrm{exc}} \to \mathcal{A}_P$ is a homotopy equivalence.* $\quad\square$

(2.3) The Borel-Serre compactification and its quotients. *The spaces $\overline{X}$ and $\overline{X}^{\mathrm{red}}$ below are defined the same way in the absence of Hermitian structure.*

The reader may assume again, for simplicity only, that G is simple over $\mathbb{Q}$, for the general group is almost a product of these. Let R be any parabolic subgroup (not necessarily maximal), and S_R the maximal $\mathbb{Q}$-split torus of the center of R^{red}. Then A_R, the identity component of $S_R(\mathbb{R})$, acts *geodesically* on D [BS:§3]. (In the case where $G = SL(2)$, D is the upper half-plane, and P is the group of upper-triangular matrices, A_P is the the group of diagonal matrices, and the A_P-orbits are the vertical lines.) When we write R as in (2.2.11), we have

$$(2.3.1) \qquad\qquad A_R = \prod_{P \in \mathcal{S}} A_P.$$

The simple roots in $\mathcal{S}$ set up an isomorphism $A_R \simeq (0, \infty)^{\mathcal{S}}$, which one uses to define $\overline{A}_R$ as $(0, \infty]^{\mathcal{S}}$, and then the corner $D(R) = D \times^{A_R} \overline{A}_R$. The Borel-Serre boundary face associated to R is

$$e(R) \simeq D \times^{A_R} \{(\infty, ..., \infty)\} \subset D(R);$$

it decomposes (as a manifold) as

$$(2.3.2) \qquad e(R) \simeq D/A_R \simeq e(R)^{\mathrm{red}} \times W_R(\mathbb{R}); \quad \text{also} \quad D \simeq e(R) \times A_R.$$

Here $e(R)^{\mathrm{red}}$ is the "symmetric space" (with Euclidean factors allowed) of L_R/A_R (cf. $D_{\ell,P}$ in (2.2)). It is easy to see that $D(R)$ contains $e(Q)$ for all $Q \supset R$. Then with a natural topology,

$$\overline{D} = \bigcup \{D(R) : R \text{ is parabolic in } G\} = D \sqcup \bigsqcup \{e(R) : R \text{ is parabolic in } G\}$$

is a manifold-with-corners with the $e(R)$'s as its open faces [BS:§7]. The closure $\overline{e(R)}$ of $e(R)$ in $\overline{D}$ adjoins to $e(R)$ exactly those $e(Q)$ with $Q \subseteq R$; indeed, the Borel-Serre construction actually applies to $e(R)$, and then $\overline{e(R)}$ is *its* Borel-Serre compactification. We have

$$(2.3.3) \qquad\qquad \overline{e(R)} = \bigcap \{\overline{e(P)} : P \in \mathcal{S}\}.$$

There is an evident $G(\mathbb{Q})$-action on $\overline{D}$, such that for $g \in G(\mathbb{Q})$, $g \cdot e(R) = e(gRg^{-1})$. Given any neat arithmetic subgroup Γ of $G(\mathbb{Q})$, one puts $\overline{X} = \Gamma\backslash\overline{D}$;

this is the *Borel-Serre compactification* of X. It is a compact manifold-with-corners with open faces of the form

$$e'(R) \simeq \Gamma(R)\backslash e(R),$$

and these are parametrized by Γ-conjugacy classes of parabolic subgroups R of G. From (2.3.2), one sees that $e'(R)$ is a fiber bundle over $e'(R)^{\mathrm{red}} =: \Gamma(R)\backslash e(R)^{\mathrm{red}}$ with fiber $\Gamma(W_R)\backslash W_R(\mathbb{R})$, a compact nilmanifold. As was the case for the toroidal boundary in (2.2),

$$\bigcap \{\overline{e'(P)} : P \in \mathcal{S}\}$$

has the same finite number of connected components (arising for the same reason as (2.2.17)), one of which is $\overline{e'(R)}$.

We will be working with two natural quotients of $\overline{X}$, $\overline{X}^{\mathrm{exc}}$ and $\overline{X}^{\mathrm{red}}$, the *excentric* and *reductive* Borel-Serre compactifications of X resp. (The space $\overline{X}^{\mathrm{red}}$ played a central role in our previous article [Z4], where it was denoted M^{RBS}.) They are determined by the respective quotients of the boundary strata:

$$(2.3.4) \qquad e(R) \to U_P(\mathbb{R})\backslash e(R) =: e(R)^{\mathrm{exc}} \to e(R)^{\mathrm{red}},$$

when R is subordinate to the maximal parabolic P. As such, all three spaces have the same number of boundary strata, indeed the same number as X^{tor}. They fit into a tower of compactifications:

$$(2.3.5) \qquad \overline{X} \to \overline{X}^{\mathrm{exc}} \to \overline{X}^{\mathrm{red}} \to X^*.$$

That the Baily-Borel compactification X^* is a quotient of $\overline{X}^{\mathrm{red}}$ is shown in [Z3:(3.11)] and [GT:2.6.3]. The quotient mapping $\overline{X}^{\mathrm{red}} \to X^*$ is given by the following. Suppose that R is subordinate to the maximal parabolic P. Then there is a product decomposition

$$(2.3.6) \qquad R^{\mathrm{red}} = (G_{\ell,R})^{\mathrm{red}} \times G_{h,P},$$

with $G_{\ell,R}$ as in (2.2.12). This induces a projection

$$(2.3.7) \qquad e(R)^{\mathrm{red}} \to D_P, \qquad \text{and thereby} \qquad e'(R)^{\mathrm{red}} \to M_P.$$

Letting R vary over parabolics subordinate to P, one gets (cf. [Z2: p. 351]):

Proposition 2.3.8. *Let $D_{\ell,P}$ be the space of type $S-\mathbb{Q}$ associated to $\widehat{G}_{\ell,P}$ (introduced after (2.2.8)), and $X_{\ell,P}$ the quotient of $D_{\ell,P}$ by $\Gamma(G_{\ell,P})$. Let $\overline{\pi}^{\mathrm{red}} : \overline{X}^{\mathrm{red}} \to X^*$ be the natural mapping. For $x \in M_P \subset X^*$,*

$$(\overline{\pi}^{\mathrm{red}})^{-1}(x) \simeq (\overline{X_{\ell,P}})^{\mathrm{red}},$$

the reductive Borel-Serre compactification of $X_{\ell,P}$. $\square$

In analogy with (2.3.6), we write

(2.3.9) $$R^{\mathrm{exc}} = (R_{\ell,P}) \times G_{h,P} \times V_P,$$

with $V_P = W_P/U_P$ again. It is easy to see that the following analogue of Proposition 2.3.8 holds:

Proposition 2.3.10. *With notation as in* Proposition 2.3.8, *let $\overline{\pi}^{\mathrm{exc}} : \overline{X}^{\mathrm{exc}} \to X^*$ be the natural mapping. For $x \in M_P \subset X^*$, $(\overline{\pi}^{\mathrm{exc}})^{-1}(x)$ is a $\Gamma(V_P)$-fibration over $\overline{X_{\ell,P}}$.* $\square$

It is the central issue in this article that, except in easy cases (viz., when G is of $\mathbb{Q}$-rank zero or $\mathbb{R}$-rank one), the tower (2.3.5) is incompatible with the tower (2.2.18), in that there are no morphisms of compactifications of X, in either direction, between a Borel-Serre and a toroidal compactification of any sort (plain, excentric, or reductive). It was shown by L. Ji that $\mathrm{GCQ}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) = X^*$ [J]. In [GT], continuous mappings $X^{\mathrm{tor}} \to \overline{X}^{\mathrm{red}}$ are constructed that are homotopy equivalent to a morphism of compactifications, and we will be elaborating on that in §3.

Suppose that a discrete group Θ acts continuously on a space Y. Recall that Borel defined a homotopy class of spaces $\langle \Theta, Y \rangle$ that is realized by $\Theta \backslash Z$ for any free Θ-space Z mapping Θ-equivariantly to Y. This is known as the *Borel construction*. It is at the heart of the notion of equivariant cohomology. From (2.3.9) and Corollary 2.2.20, we deduce:

Proposition 2.3.11. *The excentric boundary strata $\overset{\circ}{Z}_R^{\mathrm{exc}}$ and $e'(R)^{\mathrm{exc}}$ are homotopy equivalent, both giving models for the Borel construction $\langle \Gamma(G_{\ell,R}), \mathcal{A}_P \rangle$ for the action of $\Gamma(G_{\ell,R})$ on $\mathcal{A}_P$.* $\square$

(2.3.12) *Remark.* The analogous statement can be seen to hold for the closed strata, i.e., Z_R^{exc}, $\overline{e'(R)^{\mathrm{exc}}}$, and $\langle \Gamma(G_{\ell,R}), \overline{\mathcal{A}_P}^{\mathrm{exc}} \rangle$. Such facts are suggestive of (3.5.11) below.

We conclude with a curious assertion that holds in the non-Hermitian case as well. Any *Satake compactification* X^{Sa} of X (in the sense of [Z3]) is a quotient of $\overline{X}^{\text{red}}$ [Z3:(3.11)]. An instance of this is $X^{Sa} = X^*$, as in (2.3.5). We have morphisms of compactifications of X:

$$(2.3.13) \qquad\qquad \overline{X} \to \overline{X}^{\text{red}} \to X^{Sa},$$

which is the Γ-quotient of partial compactifications of D:

$$(2.3.14) \qquad\qquad \overline{D} \to \overline{D}^{\text{red}} \to D^{Sa}.$$

If we assume that Γ is neat, we have that Γ acts freely on $\overline{D}$. From [Z3], one sees that the fibers of the mappings in (2.3.14) are contractible, so these mappings are homotopy equivalences (indeed, as $\overline{D}$ is contractible, so are the other two spaces). We need say no more (compare [HZ1:(3.9.1)]):

Proposition 2.3.15. *Let D be a symmetric space of non-compact type, and Γ a neat arithmetically-defined group of isometries of D. Then $\overline{X}$ is a model for the Borel constructions $\langle \Gamma, \overline{D}^{\text{red}} \rangle \approx \langle \Gamma, D^{Sa} \rangle$.* $\square$

(2.A) *Appendix to* §2: **Hybrid compactifications.** We wish to bring work of L. Ji [J] into the current context. When X is a locally symmetric variety, the spaces X^{tor} and $\overline{X}$ are quite different in general, as we mentioned at the end of (2.3). Ji showed that their greatest common quotient is a topological space he called the *intermediate compactification*, which we will notate as X^J, that differs from X^* only in its highest-dimensional boundary stratum. An easier, but less detailed version of that is the one intended in [HZ2: Conj. (1.5.8)]: the greatest common quotient of X^{tor} and $\overline{X}^{\text{red}}$ is X^*.

We first present an auxiliary construction, namely the determination of certain "group-theoretic" compactifications between $\overline{X}^{\text{exc}}$ and $\overline{X}^{\text{red}}$. This gets decided on $\overline{D}^{\text{exc}}$, so we consider the quotient mappings $e(R)^{\text{exc}} \to e(R)^{\text{red}}$ for all R. We seek to determine whether one gets a Hausdorff space by contracting the fibers for only certain R; to give them a name, we call such spaces *hybrid compactifications*. Taking the question back to $\overline{D}$, we will use the criterion implicit in [Z1]:

(2.A.1) Condition. *For each R, let $\Pi(R) = P$, and take $\widehat{W}_R$ to be one of W_R or U_P. Assume that this is done compatibly with $G(\mathbb{Q})$-conjugacy: $\widehat{W}_{gRg^{-1}} = g(\widehat{W}_R)g^{-1}$. Then the following condition must be satisfied: $\widehat{W}_R \subseteq \widehat{W}_{R'}$ whenever $R' \subset R$.*

Remark. The above is satisfied, of course, for the choice $\widehat{W}_R = W_R$ for all R, or when $\widehat{W}_R = U_P$ for all R.

We make use of the following simple lemma, which we state without proof:

(2.A.2) Lemma. *Let $R' \subset R$ be parabolic, with $\Pi(R) = P$ and $\Pi(R') = P'$. Then*

   i) $U_P \subseteq W_{R'}$ always,

   ii) $W_R \subseteq U_{P'}$ only if both $R = P' = P$ and $W_P = U_P$.   □

From the preceding lemma, it follows that aside from $\overline{X}^{\mathrm{red}}$ there are only r distinct hybrid compactifications of the above sort: for each maximal parabolic type Q, put $\widehat{W}_R = W_R$ if and only if $\Pi(R) \prec Q$, and call the space $X^{\prec Q}$. (Note that we get $\overline{X}^{\mathrm{exc}}$ when Q is smallest with respect to "$\prec$".) These fit into a tower, with a mapping $X^{\prec Q} \to X^{\prec P}$ whenever $Q \prec P$. We have been leading up to:

Proposition 2.A.3. *When Q is the largest maximal parabolic type, there is a Cartesian square*

$$
\begin{array}{ccc}
X^{\prec Q} & \longrightarrow & X^J \\
\downarrow & & \downarrow \\
\overline{X}^{\mathrm{red}} & \longrightarrow & X^*
\end{array}
$$

□

3. The least common modification of $\overline{X}^{\mathrm{exc}}$ and $X^{\mathrm{tor,exc}}$

In this Section, we will give an elaboration on a theorem of [GT], which asserts that the natural mapping $\mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) \to X^{\mathrm{tor}}$ has contractible fibers (so is a homotopy equivalence). We make systematic use of the language of §1.

(3.1) Main result and elements of the proof. Our present goal is to prove:

Theorem 3.1.1. *The canonical morphism $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}$, having contractible fibers, is a homotopy equivalence.*

We will then see (Corollary 3.5.13, (ii)) that the corresponding assertion (Goresky-Tai) for $\mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) \to X^{\mathrm{tor}}$ is deducible from Theorem 3.1.1 via LCM-basechange (1.1.8).

To begin, we have from §2 that the spaces $\overline{X}^{\mathrm{exc}}$ and $X^{\mathrm{tor,exc}}$ are stratified, and have X^* as a (stratified) common quotient. By Proposition 1.1.5, the assertion of Theorem 3.1.1 can get decided locally over X^*. In both cases, the fibers of the natural mapping to X^* over the stratum M_P involve V_P and the homogeneous

cone C_P from (2.2), through the boundaries of partial compactifications of $C'_P :=$ $\Gamma(G_{\ell,P})\backslash C_P$ (see (3.5) below).

Let $\check{D}(P)$ denote the open subset $U_P(\mathbb{C})\cdot D$ of the compact dual of D (cf. (2.2.1)). In [AMRT: III, pp. 235–236, 250] we find the following, which allows us to examine the excentric Borel-Serre and toroidal cases simultaneously:

Proposition 3.1.2. *With P acting on C_P and D_P through P^{red}, the canonical P-equivariant decomposition*

$$(3.1.2.1) \qquad\qquad U_P(\mathbb{R})\backslash D \simeq C_P \times D_P \times V_P(\mathbb{R})$$

extends to a P-equivariant decomposition

$$(3.1.2.2) \quad U_P(\mathbb{R})\backslash \check{D}(P) \simeq U_P(\mathbb{R})^{\mathrm{im}} \times [U_P(\mathbb{C})\backslash\check{D}(P)] \simeq U_P(\mathbb{R})^{\mathrm{im}} \times D_P \times V_P(\mathbb{R}),$$

where "im" reminds us that this copy of $U_P(\mathbb{R})$ is represented by the imaginary part of $U_P(\mathbb{C})$. $\square$

It follows that the excentric quotients on $\overline{X}$ and X^{tor} are induced by constructions on (3.1.2.2), after which one must undo the $U_P(\mathbb{R})$-quotients in the interior.

Underlying Proposition 3.1.2 are the facts that $G_{h,P}$ centralizes U_P, and that the change of basepoint for the Cayley transform can be effected by translation by an element of $U_P(\mathbb{R})^{\mathrm{im}}$ (which is a consequence of the same for $G = SL_2$). In effect, when working over the Baily-Borel boundary stratum M_P, after acknowledging the role of $V_P(\mathbb{R})$, we can "replace" X by C_P and determine the least common boundary modification (see (1.1.3)) of the two partial compactifications of the cone.

We will also make use of the fact mentioned in (2.3) that $\Gamma(G_{\ell,P})\backslash\widehat{\Sigma}_P$ is a Satake compactification of $X_{\ell,P}$, so is in particular a quotient of $\overline{X}_{\ell,P}^{\mathrm{red}}$ [Z3:(3.11)].

(3.2) Real and complex toroidal embeddings; dual compactifications. A spanning simplicial cone σ, as in (1.2), determines a smooth (complex) affine torus embedding:

$$(3.2.1) \qquad\qquad T \simeq (\mathbb{C}^*)^d \subset \mathbb{C}^d \simeq T_\sigma$$

in the usual manner (see [AMRT: I,§1]). As described in [HZ1,§2], σ also determines a real torus embedding $T_{\mathbb{R};\sigma}$. The latter admits two descriptions: it is the quotient of T_σ by the maximal compact torus of T (viz., $T^c \simeq (S^1)^d$), and it is represented by the closure of $(\mathbb{R}^+)^d$ in T_σ. Then $\sigma \subset T_{\mathbb{R};\sigma} \simeq (\mathbb{R}^{\geq 0})^d$ is given by $(0,1]^d$ (see (1.2)).

Let Σ be a fan, as in (2.2). This determines a complex torus embedding T_Σ by gluing the T_σ's, and likewise a real torus embedding $T_{\mathbb{R};\Sigma}$. Give $|\Sigma|$ the largest topology for which the inclusion mappings of its cones are continuous. We then define two partial compactifications, $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$, by attaching to Σ boundaries induced by the constructions in (1.2) on each simplex; with $\Sigma \simeq \widehat{\Sigma} \times (0, \infty)$,

$$(3.2.2) \qquad \overline{\Sigma}_1 = \bigcup \{\overline{\sigma}_1 : \sigma \in \Sigma\} \qquad \text{and} \qquad \overline{\Sigma}_2 = \bigcup \{\overline{\sigma}_2 : \sigma \in \Sigma\},$$

likewise with the corresponding topology for the inclusions $\overline{\sigma}_i \hookrightarrow \overline{\Sigma}_i$. Also, put

$$(3.2.3) \qquad \widehat{\Sigma}_1 = \partial \overline{\Sigma}_1 \qquad \text{and} \qquad \widehat{\Sigma}_2 = \partial \overline{\Sigma}_2.$$

Then both $\widehat{\Sigma}_1$ (plainly) and $\widehat{\Sigma}_2$ (cf. (1.2.4)) are homeomorphic to $\widehat{\Sigma}$.

Lemma 3.2.4. *For any $\widehat{\tau} \in \widehat{\Sigma}$, let $\tau^\vee(\widehat{\sigma})$ be as in* Proposition 1.2.1, *and put*

$$(3.2.4.1) \quad \tau^\vee = \bigcup \{\tau^\vee(\widehat{\sigma}) \,|\, \widehat{\sigma} \text{ is a top-dimensional simplex in } \widehat{\Sigma} \text{ that contains } \widehat{\tau}\}.$$

Then $\tau^\vee$ is contractible.

Proof. In the notation of (1.2), we have that for $\tau = \tau_J \subseteq \sigma$, τ is given by the equations $t_j = 1$ for all $j \in J$, and $\tau^\vee(\widehat{\sigma})$ is given by the equations $t_j = 0$ for all $j \notin J$. This contracts to the point that has further equations $t_j = 1$ for all $j \in J$, viz., $\widehat{\tau} \cap \tau^\vee(\widehat{\sigma})$. This point is independent of the top-dimensional cone σ (cf. [HZ1: 2.3]), so we can do the contractions simultaneously for all σ. $\square$

(3.2.5) *NB*—The union in (3.2.4.1) can be infinite if $\widehat{\tau}$ is in the boundary of $\widehat{\Sigma}$.

Recall the notion of LCMb that was defined in (1.1.3). We have the following extension of Proposition 1.2.1, which we write for $\Sigma = \Sigma_P$:

Proposition 3.2.6 (Canonical duality). $\mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = \bigcup \{\widehat{\tau} \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P\}$.

(3.2.7) *Remark.* The formula in Proposition 3.2.6 can be rewritten as

$$(3.2.7.1) \qquad \mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = \bigsqcup \{\widehat{\tau}^\circ \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P\},$$

for $\tau^\vee \subset \omega^\vee$ when $\widehat{\omega}$ is a face of $\widehat{\tau}$. Also, we note that the interior of $\widehat{\Sigma}_1$ can be written as

$$(3.2.7.2) \qquad \widehat{C}_P = \bigsqcup \{\widehat{\tau}^\circ : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}.$$

In the above, $\widehat{\tau}^\circ$ denotes the interior of the simplex $\widehat{\tau}$. We can switch the roles in (3.2.7.1) and write:

$$(3.2.7.3) \qquad \mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2) = \bigcup \{\widehat{\tau} \times (\tau^\vee)^\circ : \widehat{\tau} \in \widehat{\Sigma}_P\}.$$

Proof of Proposition 3.2.6. Let $\{x_j\}$ be a sequence in Σ that converges in both $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$. Suppose that the limit in $\widehat{\Sigma}_1$ lies in the interior $\widehat{\tau}^\circ$ of the simplex $\widehat{\tau}$. We claim that there is a top-dimensional cone σ containing τ (as a face) such that σ contains a subsequence of $\{x_j\}$ (because there can be infinitely many such σ, there is something to check). If not, each such σ contains only finitely many x_j's. Thus, there is a neighborhood $N_\tau(\sigma)$ of τ in σ that contains no x_j's. By the definition of the topology, $\bigcup_\sigma N_\tau(\sigma)$ is open in Σ, and it contains no x_j's. This contradicts the convergence. It follows that we may assume that the sequence is in a single σ, and we are reduced to Proposition 1.2.1: the second limit lies in $\tau^\vee(\widehat{\sigma})$. Since this happens for every $\sigma \supset \tau$, we use (3.2.4.1) and our formula for $\mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2)$ follows. $\square$

Suppose next that a discrete group Γ_ℓ acts linearly on a cone complex Σ in a separated and discontinuous manner, such that there are only finitely many Γ_ℓ-equivalence classes of cones. We use "prime" to indicate a Γ_ℓ-quotient, as in $\Sigma' = \Gamma_\ell \backslash \Sigma$, and do likewise in (3.2.2) to obtain

$$(3.2.8) \qquad \overline{\Sigma}_1' = \Gamma_\ell \backslash \overline{\Sigma}_1 \qquad \text{and} \qquad \overline{\Sigma}_2' = \Gamma_\ell \backslash \overline{\Sigma}_2.$$

The spaces in (3.2.8) are viewed as compactifications of Σ'. Let $\widehat{\Sigma}$ be the corresponding simplicial complex, as in (2.2). Then we have $\widehat{\Sigma}' = \Gamma_\ell \backslash \widehat{\Sigma}$, a compact space. From (3.2.3) and (3.2.6) we get a pair of polyhedral models of $\widehat{\Sigma}'$ having finitely many faces:

$$(3.2.9) \qquad \widehat{\Sigma}_1' = \Gamma_\ell \backslash \widehat{\Sigma}_1 \qquad \text{and} \qquad \widehat{\Sigma}_2' = \Gamma_\ell \backslash \widehat{\Sigma}_2,$$

with the latter given as in (1.2.4).

Since Γ_ℓ acts on $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$, it acts of their LCM, i.e. the LCMb in Proposition 3.2.6. The expected conclusion holds to a large extent:

Proposition 3.2.10. *If Γ_ℓ is neat, the actions of Γ_ℓ on $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ are diagonal, and*

$$(3.2.10.1) \qquad \mathrm{LCMb}(\widehat{\Sigma}_1', \widehat{\Sigma}_2') = \Gamma_\ell \backslash \bigcup \{\widehat{\tau} \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}\}.$$

Proof. By hypothesis, there is a subset S of Σ, the union of finitely many closed cones, such that the mapping $S \to \Sigma \to \Sigma'$ is surjective. This gives the same for the respective closures of S in $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$:

$$\overline{S}_1 \to \overline{\Sigma}_1 \qquad \text{and} \qquad \overline{S}_2 \to \overline{\Sigma}_2.$$

Moreover, $\overline{S}_1$ and $\overline{S}_2$ are compact. Thus, we are in the setting of Proposition 1.3.8, provided we verify that the actions of Γ_ℓ on $\overline{\Sigma}_1$ and $\overline{\Sigma}_2$ are diagonal.

Let $\widehat{\tau} \in \widehat{\Sigma}_1 = \partial\overline{\Sigma}_1$, $e_1 \in \widehat{\tau}$ and $e_2 \in (\tau^\vee)^\circ$. As $\widehat{\tau}$ varies, this gives all points (e_1, e_2) of $\mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2)$, by (3.2.7.3). Suppose that $(\gamma \cdot e_1, e_2) \in \mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2)$ for some $1 \neq \gamma \in \Gamma_\ell$. A priori, there are two cases. If $\gamma \cdot \widehat{\tau} = \widehat{\tau}$, then γ acts as a permutation of the vertices of $\widehat{\tau}$, so it fixes the barycenter of $\widehat{\tau}$. Since Γ_ℓ us assumed neat, γ must fix the whole boundary component containing τ°, hence also its closure. In particular, $\gamma \cdot e_1 = e_1$. On the other hand, suppose that $\gamma \cdot \widehat{\tau} \neq \widehat{\tau}$. The only way that both (e_1, e_2) and $(\gamma \cdot e_1, e_2)$ could be in the LCMb is if $e_2 \in \tau^\vee \cap (\gamma \cdot \tau)^\vee$. However, this set would lie in the *boundary* of $\tau^\vee$, a contradiction. The conclusion is, then, that $\gamma \cdot e_1 = e_1$, so the actions are diagonal.

This gives that the canonical mapping

$$\Gamma_\ell \backslash \mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2) \to \mathrm{LCMb}(\widehat{\Sigma}'_1, \widehat{\Sigma}'_2)$$

is an isomorphism. We use Proposition 3.2.6 to obtain formula (3.2.10.1). $\square$

(3.2.10.2) *Remarks.* i) One might try to prove Proposition 3.2.10 by using (3.2.7.1) instead of (3.2.7.3). The reader is invited to investigate why we took the other route.

ii) If $\widehat{\tau}$ above is just the vertex $\{e_1\}$, the situation is very much like that in (1.3.6), but here we keep Γ_ℓ fixed.

(3.3) Adjustments to duality. For a pair of reasons, $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}})$ cannot be determined from Proposition 3.2.10 alone. First, the projection (from Proposition 3.1.2) onto $\widehat{C}'_P$ of the P-stratum of $X^{\mathrm{tor,exc}}$ is not dense in $\widehat{\Sigma}'_2$; it is only the (compact) $\Gamma(G_{\ell,P})$-quotient of the subcomplex

$$(3.3.1) \qquad\qquad (\widehat{\Sigma}^c_P)^\vee = \bigcup \{\tau^\vee : \widehat{\tau} \in \widehat{\Sigma}^c_P\},$$

which is contained in the interior of $\widehat{\Sigma}_2$. Then $(\widehat{\Sigma}^c_P)^\vee$ can be identified, via (1.2.4), with $\widehat{\Sigma}^{(1)}_P$. Here, the barycentric subdivision $\Sigma^{(1)}_P$ of Σ_P also gives a compatible collection of fans as P varies. In Corollary 3.3.5 below, we show that

$$(3.3.3) \qquad\qquad (\widehat{\Sigma}^c_P)^\vee \xrightarrow{\sim} (\widehat{\Sigma}^\circ_P)^\vee,$$

where $\widehat{\Sigma}_P^\circ$ denotes the $\Gamma(G_{\ell,P})$-invariant subset of $\widehat{\Sigma}_P$ given by (2.2.9).

Looking from the inside, we have that going to infinity in $\overline{X}$ occurs not just by letting the dilation variable of the cone, relative to a cross-section of Σ_P, go to ∞. Rather, it must go to infinity at least as fast as a so-called *core* for the cone,—see [AMRT: II, §5] for the notion of a core—which gives a cross-section of $C_P \subset \Sigma_P$ that blows up at the boundary cones of Σ_P (the precise meaning of a core is irrelevant for this article). One gets something essentially the same by using the "$G_{\ell,P}$ side" of the description in [Z2: 3.18].

We take a brief excursion into the calculus of joins. Let $\widehat{\omega}$ be the face of a simplex $\widehat{\sigma}$ that is opposite the face $\widehat{\tau}$ of $\widehat{\sigma}$, i.e., $\widehat{\omega}$ is spanned by the vertices of $\widehat{\sigma}$ that are *not* in $\widehat{\tau}$. Then $\widehat{\sigma}$ is the *join* $\widehat{\omega} * \widehat{\tau}$, namely the quotient of $\widehat{\omega} \times [0,1] \times \widehat{\tau}$ obtained by separately collapsing $\widehat{\omega} \times \{1\}$ to a point and $\{0\} \times \widehat{\tau}$ to a point. The face $\widehat{\omega}$ of $\widehat{\sigma}$ lies at $s = 0$ and $\widehat{\tau}$ is at $s = 1$, where s denotes the variable of $[0,1]$. One can collapse $\widehat{\sigma} - \widehat{\omega}$, and also its subset $\tau^\vee(\sigma)$, onto $\widehat{\tau}$ by the following deformation retraction: $\{h_s\}$ is given by mappings on the product $\widehat{\omega} \times (0,1] \times \widehat{\tau}$ that depend only on the interval variable: $f_s(t) = \min\{t + s, 1\}$; $h_s(w, t, x) = (w, f_s(t), x)$. Note that $\tau^\vee(\sigma)$ is closed under the flow.

The setting above is for disjoint faces of one simplex in a complex. However, the notion can easily be extended to intersecting faces. Let $\widehat{\tau}$ and $\widehat{\omega}$ be simplices with intersection $\widehat{\alpha}$. Write $\widehat{\tau} = \widehat{\alpha} * \widehat{\tau}'$ and $\widehat{\omega} = \widehat{\alpha} * \widehat{\omega}'$. Then $\widehat{\tau} * \widehat{\omega} = \widehat{\tau} * \widehat{\omega}' = \widehat{\tau}' * \widehat{\omega} = \widehat{\alpha} * \widehat{\tau}' * \widehat{\omega}'$, the smallest simplex containing both $\widehat{\tau}$ and $\widehat{\omega}$. We are interested in the dual, in the sense of (3.2.4.1), of a join.

Proposition 3.3.4. *i) Let $\widehat{\tau}$ and $\widehat{\omega}$ be faces of a top-dimensional simplex $\widehat{\sigma}$ in a simplicial complex $\widehat{\Sigma}$. Then*

$$(3.3.4.1) \qquad (\widehat{\tau} * \widehat{\omega})^\vee(\widehat{\sigma}) = \tau^\vee(\widehat{\sigma}) \cap \omega^\vee(\widehat{\sigma}).$$

ii) Conversely, if $\widehat{\tau}$ and $\widehat{\omega}$ are simplices in $\widehat{\Sigma}$ such that $\tau^\vee \cap \omega^\vee \neq \emptyset$, then $\widehat{\tau} * \widehat{\omega}$ is defined and

$$(3.3.4.2) \qquad (\widehat{\tau} * \widehat{\omega})^\vee = \tau^\vee \cap \omega^\vee.$$

Proof. We use the notation from the proof of Proposition 1.2.1. If we write $\widehat{\tau} = \widehat{\tau}_J$ and $\widehat{\omega} = \widehat{\tau}_K$, then $\widehat{\tau} * \widehat{\omega} = \widehat{\tau}_{(J \cap K)}$. Similarly, we have $\tau^\vee = \tau_J^\vee$, with defining equations parametrized by the *complement* of J, and likewise for K, so $\tau_J^\vee \cap \tau_K^\vee = \tau_{(J \cap K)}^\vee$. This gives i).

As for ii), because more than one top-dimensional simplex is involved in the dual, we must be a little careful. From the definition, we have

$$(3.3.4.3) \quad \tau^\vee \cap \omega^\vee = \bigcup \{\tau^\vee(\widehat{\sigma}) \cap \omega^\vee(\widehat{\sigma}') : \widehat{\sigma}, \widehat{\sigma}' \text{ top-dimensional simplices in } \widehat{\Sigma}\}.$$

There is a natural notion of $\tau^\vee(\widehat{v})$ for any simplex $\widehat{v} \supset \widehat{\tau}$: for any top-dimensional $\widehat{\sigma} \supset \widehat{v}$, take $\tau^\vee(\widehat{\sigma}) \cap \widehat{v}$, and check that this is independent of the choice of $\widehat{\sigma}$. Then $\tau^\vee(\widehat{\sigma}) \cap \omega^\vee(\widehat{\sigma}') = (\tau^\vee \cap \omega^\vee)(\widehat{\sigma} \cap \widehat{\sigma}')$. We see that we can then rewrite (3.3.4.3) as

$$(3.3.4.4) \quad \tau^\vee \cap \omega^\vee = \bigcup \{\tau^\vee(\widehat{\sigma}) \cap \omega^\vee(\widehat{\sigma}) : \widehat{\sigma} \text{ is a top-dimensional simplex in } \widehat{\Sigma}\}.$$

If the term in the right-hand side of (3.3.4.4) coming from $\widehat{\sigma}$ is non-empty, then both $\widehat{\tau}$ and $\widehat{\omega}$ are faces of $\widehat{\sigma}$. We can now proceed as in i), term by term. $\square$

We apply Proposition 3.3.4 to $\widehat{\Sigma}_P$:

Corollary 3.3.5. *Let $\widehat{\tau} \in \widehat{\Sigma}_P^\circ$. Then $\tau^\vee \in (\widehat{\Sigma}_P^c)^\vee$.*

Proof. Since $\widehat{\tau}$ has a vertex in $\widehat{\Sigma}_P^c$, we can write it in the form $\widehat{\alpha} * \widehat{\omega}$, with $\widehat{\alpha} \in \widehat{\Sigma}_P^c$. It follows from (3.3.4, ii) that $\tau^\vee \subseteq \alpha^\vee$. $\square$

We remember this fact as the equivalent (3.3.3).

With that said, the calculations in (3.2) get used as follows. We now regard $\overline{\Sigma}_1$ as an element of $\mathfrak{PCp}(C_P)$. Let $(\Sigma_P)^{\text{tor,exc}}$ be the partial compactification of C_P contained in $\overline{\Sigma}_2$ with $\partial(\Sigma_P)^{\text{tor,exc}} = (\widehat{\Sigma}_P^c)^\vee \subset \widehat{\Sigma}_2$. We consider

$$(3.3.5.1) \qquad \text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) = \partial\text{LCM}(\overline{\Sigma}_1, (\Sigma_P)^{\text{tor,exc}}) \subset \text{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2),$$

in which the rightmost space is given by (3.2.7.3). If $\widehat{\tau} \in \widehat{\Sigma}_P^\circ$, then $\tau^\vee$ is interior to $\partial\overline{\Sigma}_2$ by Corollary 3.3.5, and thus the contributions of $\widehat{\tau}$ to $\text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$ and $\text{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2)$ are the same. At issue, then, are the simplices $\widehat{\tau}$ contained in the boundary of $\widehat{\Sigma}_P$.

We see from (3.2.7.1) and (3.2.7.2) that the subset of $\text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$ that maps to $\widehat{C}_P$ under the projection onto $\widehat{\Sigma}_1$ is given by

$$(3.3.6) \qquad\qquad \mathcal{T} := \bigsqcup \{\widehat{\tau}^\circ \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}.$$

$\mathcal{T}$ visibly maps *onto* $(\widehat{\Sigma}_P^c)^\vee$ under the projection $\text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to (\widehat{\Sigma}_P^c)^\vee$. Therefore, $\text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$ necessarily contains the closure $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ in $\widehat{\Sigma}_1 \times (\widehat{\Sigma}_P^c)^\vee$:

$$(3.3.7) \qquad\qquad \text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \supseteq \widetilde{\mathcal{T}} = \bigcup \{\widehat{\tau} \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}.$$

The following determination is fundamental:

Proposition 3.3.8. $\mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$ *is equal to* $\widetilde{\mathcal{T}}$.

Proof. One must keep in mind that $\widehat{\Sigma}_P^\circ$ is not a complex. Let $\widehat{\omega}$ be a simplex in $\widehat{\Sigma}_P$. (We are thinking that $\widehat{\omega}$ is in the boundary of $\widehat{C}_P$, but it does not matter.) In canonical duality this contributes, in (3.2.7.1), $\widehat{\omega}^\circ \times \omega^\vee$ to $\mathrm{LCMb}(\widehat{\Sigma}_1, \widehat{\Sigma}_2)$; in $\mathrm{LCMb}(\widehat{\Sigma}_1, (\Sigma_P^c)^\vee)$ the contribution is

$$(3.3.8.1) \qquad\qquad \widehat{\omega}^\circ \times (\omega^\vee \cap (\Sigma_P^c)^\vee).$$

Suppose that $\widehat{\tau} \in \widehat{\Sigma}_P^c$ and $\omega^\vee \cap \tau^\vee \neq \emptyset$. By Proposition 3.3.4, ii), $\widehat{\omega} * \widehat{\tau}$ is defined and its dual is $(\widehat{\omega} * \widehat{\tau})^\vee = \tau^\vee \cap \omega^\vee$. Since $\widehat{\omega} * \widehat{\tau} \in \widehat{\Sigma}_P^\circ$, we get that

$$(3.3.8.2) \qquad\qquad \widehat{\omega}^\circ \times (\omega^\vee \cap \tau^\vee) \subset (\widehat{\omega} * \widehat{\tau}) \times (\widehat{\omega} * \widehat{\tau})^\vee.$$

As $(\omega^\vee \cap \tau^\vee) \in (\widehat{\Sigma}_P^c)^\vee$ by Corollary 3.3.5, we are done. $\square$

Thus, the first adjustment to canonical duality is to replace $\widehat{\Sigma}_P$ by $\widehat{\Sigma}_P^\circ$ in the formula in Proposition 3.2.6 (see (3.3.11.2) below.)

The second one can now be handled rather quickly. Given the cone $C = C_P$, let $s : \widehat{C} \to C$ be any cross-section of the cone. Here we do not assume that s extends to a cross-section of Σ_P. One uses s to write

$$(3.3.9) \qquad\qquad C \simeq \widehat{C} \times (0, \infty).$$

This again determines variables $(r, \widehat{x})$ on C, such that $x = r \cdot s(\widehat{x})$ for all $x \in C$. We can define $\overline{C}(s)$ in terms of (3.3.9) as $\widehat{C} \times (0, \infty]$ (compare the definition of $\overline{\sigma}_1$ in (1.2)). Note that if $s_2 > s_1$, then convergence to infinity in $\overline{C}(s_2)$ implies the same in $\overline{C}(s_1)$, and the limits coincide in $\widehat{C} \times \{\infty\}$.

Take s to be a section determined by a $\Gamma(G_{\ell,P})$-invariant core of C, which always exists [AMRT: p.123]. The stronger notion of convergence in $\overline{C}(s)$ than in $\overline{\Sigma}_1$ implies that

$$(3.3.10) \qquad \mathrm{LCM}(\overline{C}(s), (\Sigma_P)^{\mathrm{tor,exc}}) \subseteq \mathrm{LCM}(\overline{\Sigma}_1, (\Sigma_P)^{\mathrm{tor,exc}})$$

But the set of pairs of limits we get, using $\widehat{\Sigma}_1$, is proved in Proposition 3.3.8 to be the least possible, so it must be the same set here. It follows that we have equality in (3.3.10); the fact that s blows up at the boundary of C is irrelevant for the determination of the LCM. In other words, the second adjustment to canonical duality is vacuous. In sum,

Proposition 3.3.11. *Let s be a $\Gamma(G_{\ell,P})$-invariant cross-section of C that blows up at the boundary of Σ_P. Then*

$$(3.3.11.1) \qquad \mathrm{LCMb}(\widehat{C}(s), (\Sigma_P^c)^\vee) = \mathrm{LCMb}(\widehat{\Sigma}_1, (\Sigma_P^c)^\vee),$$

with the right-hand side given by Proposition 3.3.8 *and* (3.3.7), *i.e.:*

$$(3.3.11.2) \qquad \mathrm{LCMb}(\widehat{\Sigma}_1, (\Sigma_P^c)^\vee) = \bigcup \{\widehat{\tau} \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}.$$

In particular, this hold when s defines a core of C_P. □

(3.4) Fibers of mappings of $\Gamma(G_{\ell,P})$-quotients. We next treat the role of taking $\Gamma(G_{\ell,P})$-quotients in determining the fiber in mappings related to Theorem 3.1.1. *We assume throughout this section only that $\Gamma(G_{\ell,P})$ acts freely on $\widehat{C}_P$.*

First, we make a general and elementary observation:

Proposition 3.4.1. *Suppose that Γ acts separated and discontinuously on spaces Z_1 and Z_2. Let $\psi : Z_1 \to Z_2$ be a Γ-equivariant mapping. Let $\widetilde{F}$ be the fiber of ψ at $z \in Z_2$, and F the fiber of the induced mapping $\Gamma\backslash Z_1 \to \Gamma\backslash Z_2$ at the corresponding point $[z] \in \Gamma\backslash Z_2$. Then $F \simeq \Gamma_z\backslash\widetilde{F}$, where Γ_z is the isotropy subgroup of Γ at z.*

Proof. It is immediate that $F \simeq \Gamma\backslash(\Gamma \cdot \widetilde{F})$. The inclusion of $\widetilde{F}$ in $\Gamma \cdot \widetilde{F}$ induces $\Gamma_z\backslash\widetilde{F} \simeq F$. □

Corollary 3.4.2. *Under the conditions of* Proposition 3.4.1, *suppose that Γ acts freely on Z_2. Then $F \simeq \widetilde{F}$.* □

We use a "prime" as a standard way to denote a $\Gamma(G_{\ell,P})$-quotient, with the one exception $\widehat{\Sigma}_2'' = \Gamma(G_{\ell,P})\backslash\partial(\Sigma_P)^{\mathrm{tor,exc}}$. Because $\Gamma(G_{\ell,P})$ acts freely on $\widehat{C}_P$, the action on $(\widehat{\Sigma}_P^c)^\vee$, which is a subset of $\widehat{C}_P$, is also free. The next fact follows immediately from Proposition 3.2.10 when $\Gamma(G_{\ell,P})$ is neat. We provide a small modification of the argument to obtain some improvement of the result:

Proposition 3.4.3. *i) For $\widehat{\tau} \in \widehat{\Sigma}_P^\circ$, the fiber of $\mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to (\widehat{\Sigma}_P^c)^\vee$, at any point in the interior $(\tau^\vee)^\circ$ of $\tau^\vee$, is $\widehat{\tau}$. In particular, the fiber is contractible. Thus*

$$(3.4.3.1) \quad \mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) = \bigcup \{\widehat{\tau} \times (\tau^\vee)^\circ : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\} = \bigcup \{\widehat{\tau} \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}.$$

ii) Under the mild assumption that $\Gamma(G_{\ell,P})$ acts freely on $\widehat{C}_P$, the actions of $\Gamma(G_{\ell,P})$ on $\overline{\Sigma}_1$ and $(\Sigma_P)^{\mathrm{tor,exc}}$ are diagonal, so

$$(3.4.3.2) \qquad \mathrm{LCMb}(\widehat{\Sigma}_1', \widehat{\Sigma}_2'') = \Gamma(G_{\ell,P})\backslash\bigcup \{\widehat{\tau} \times (\tau^\vee)^\circ : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}.$$

The fiber of $\mathrm{LCMb}(\widehat{\Sigma}_1', \widehat{\Sigma}_2') \to \widehat{\Sigma}_2''$, *at any point represented by a point of* $(\tau^\vee)^\circ$, *is* $\widehat{\tau}$, *hence contractible.*

Proof. i) We note that $(\widehat{\Sigma}_P^c)^\vee$ is a polyhedral complex, whose strata are the sets of the form $(\tau^\vee)^\circ$, with $\widehat{\tau} \in \widehat{\Sigma}_P^\circ$. The analogue of (3.2.7.2), namely

$$(3.4.3.3) \qquad (\widehat{\Sigma}_P^c)^\vee = \bigcup \{(\tau^\vee)^\circ : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\},$$

holds. The fiber of the projection $\mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to (\widehat{\Sigma}_P^c)^\vee$ over $(\tau^\vee)^\circ$ is $\widehat{\tau}$ by canonical duality. (One cannot switch the roles of $\widehat{\tau}$ and $\tau^\vee$ here.) Thus:

$$(3.4.3.4) \qquad \mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) = \bigcup \{\widehat{\tau} \times (\tau^\vee)^\circ : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\};$$

taking the closure of $(\tau^\vee)^\circ$ in (3.4.3.4) effects no change.

ii) Again, there is a subset S of Σ_P, the union of finitely many closed cones, such that the mapping $S \to \Sigma_P \to \Sigma_P'$ is surjective. This gives the same for the respective closures of S in $\overline{\Sigma}_1$ and $(\Sigma_P)^{\mathrm{tor,exc}}$:

$$\overline{S}_1 \to \overline{\Sigma}_1 \qquad \text{and} \qquad S^{\mathrm{tor,exc}} \to (\Sigma_P)^{\mathrm{tor,exc}}.$$

As $\overline{S}_1$ and $S^{\mathrm{tor,exc}}$ are compact, we can then apply Proposition 1.3.8, provided we verify that diagonality holds.

Let $\widehat{\tau} \in \widehat{\Sigma}_1^\circ$, $e_1 \in \widehat{\tau}$, and $e_2 \in (\tau^\vee)^\circ$. As $\widehat{\tau}$ varies, this gives all points (e_1, e_2) of $\mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$, by (3.4.3.3). Suppose that $(\gamma \cdot e_1, e_2) \in \mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$ for some $1 \neq \gamma \in \Gamma(G_{\ell,P})$. The discussion continues almost exactly as in the proof of Proposition 3.2.10: if $\gamma \cdot \widehat{\tau} = \widehat{\tau}$, then γ fixes the barycenter of $\widehat{\tau}$. Here, though, the barycenter of $\widehat{\tau}$ is a point of $\widehat{C}_P$ because $\widehat{\tau} \in \widehat{\Sigma}_1^\circ$, in contradiction to the freeness hypothesis. Thus, $\gamma \cdot \widehat{\tau} \neq \widehat{\tau}$. This is not possible either, as in the aforementioned proof. The conclusion is that the situation described in the beginning of this paragraph does not occur, so the actions are diagonal.

This gives that the canonical mapping

$$\Gamma(G_{\ell,P}) \backslash \mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to \mathrm{LCMb}(\widehat{\Sigma}_1', \widehat{\Sigma}_2'')$$

is an isomorphism. We use (3.3.11.2) to obtain formula (3.4.3.2). By Corollary 3.4.2, the taking of $\Gamma(G_{\ell,P})$-quotients does not change the fiber, as $\Gamma(G_{\ell,P})$ acts freely on $(\widehat{\Sigma}_P^c)^\vee$. Our assertion follows. $\square$

(3.4.3.5) *Remarks.* i) Given (3.4.3.3), Proposition 3.4.3 asserts that *all* fibers of that mapping are contractible.

ii) Item i) of the proposition is to be contrasted with Proposition 3.4.12 below.

iii) The outcome of the check for diagonality in the above proof is important enough that we restate it for emphasis: *if both* (e_1, e_2) *and* $(\gamma \cdot e_1, e_2)$ $(\gamma \in \Gamma(G_{\ell,P}))$ *are in* $\mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$, *then* $\gamma = 1$.

iv) An assertion from the proof of Proposition 3.2.10, tacitly used in the above, is that $\tau^\vee \cap (\gamma \cdot \tau)^\vee \neq \emptyset$. By (3.3.4.2), there must be a simplex in $\widehat{\Sigma}_1$ (not necessarily unique) containing both $\widehat{\tau}$ and $\gamma \cdot \widehat{\tau}$ as faces, with $\tau^\vee \cap (\gamma \cdot \widehat{\tau})^\vee = (\widehat{\tau} * (\gamma \cdot \widehat{\tau}))^\vee$. However, this would not improve the argument.

We use Proposition 3.4.3 in conjunction with Proposition 3.3.11 to yield:

Corollary 3.4.4. $\mathrm{LCMb}(\widehat{C}'(s), \Sigma_2'') = \mathrm{LCMb}(\widehat{\Sigma}_1', \Sigma_2'')$. $\quad\square$

We did not really use much about $\widehat{\Sigma}_1$ in making the argument for the key Proposition 3.3.8, which underlies our calculations. The main point was that $\widetilde{\mathcal{T}}$ gave the part of the boundary contained in $\widehat{C}_P \times (\widehat{\Sigma}_P^c)^\vee$. As such, we can try to apply it likewise to other $\Gamma(G_{\ell,P})$-equivariant partial compactifications of C_P. Most interesting are ones that have familiar elements $Y \in \mathfrak{PCp}(\widehat{C}_P)$ at the boundary; writing $\widehat{C}_P$ as $D_{\ell,P}$, we have in mind $Y = \overline{D}_{\ell,P}$, $Y = \overline{D}_{\ell,P}^{\mathrm{red}}$, or Y any Satake partial compactification $D_{\ell,P}^{Sa}$ of $D_{\ell,P}$.

We need to be specific about the corresponding elements of $\mathfrak{PCp}(C_P)$. With Σ written as $\widehat{\Sigma} \times (0, \infty)$, inducing $C_P \simeq \widehat{C}_P \times (0, \infty)$, we take $Y \times (0, \infty]$ with $\partial Y \times (0, \infty)$ removed; it is stated this way because a cross-section for one Y need not work for another (compare the end of (3.3)). There is a "version" of $\widetilde{\mathcal{T}}$, i.e., the closure of (3.3.6), for each case. These we denote respectively $\widetilde{\mathcal{T}}^{BS}$, $\widetilde{\mathcal{T}}^{\mathrm{red}}$, and $\widetilde{\mathcal{T}}^{Sa}$, by which we mean

(3.4.5) $\quad \widetilde{\mathcal{T}}^{BS} = \bigcup \{\widehat{\tau}^{BS} \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}, \quad \widetilde{\mathcal{T}}^{\mathrm{red}} = \bigcup \{\widehat{\tau}^{\mathrm{red}} \times \tau^\vee : \widehat{\tau} \in \widehat{\Sigma}_P^\circ\}, \quad \text{etc.,}$

where $\widehat{\tau}^{BS}$ is the closure of $\widehat{\tau}^\circ$ in $\overline{C}_P$, $\widehat{\tau}^{\mathrm{red}}$ is the closure of $\widehat{\tau}^\circ$ in $\overline{C}_P^{\mathrm{red}}$, etc.

(3.4.6) *Remark and Convention.* We henceforth understand, when we write $D_{\ell,P}^{Sa}$ below, that the latter maps to $\widehat{\Sigma}_P$, which is itself a minimal Satake compactification of $\widehat{C}_P$ (see, e.g., [Z3:(2.10)] about morphisms of Satake partial compactifications). It would not surprise me if this turned out to be an unnecessary assumption.

One sees directly that the diagram

(3.4.7)
$$
\begin{array}{ccccccc}
\widetilde{\mathcal{T}}^{BS} & \to & \widetilde{\mathcal{T}}^{\mathrm{red}} & \to & \widetilde{\mathcal{T}}^{Sa} & \to & \widetilde{\mathcal{T}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{D}_{\ell,P} & \to & \overline{D}_{\ell,P}^{\mathrm{red}} & \to & D_{\ell,P}^{Sa} & \to & \widehat{\Sigma}_P
\end{array}
$$

is Cartesian, with fiber given by Proposition 3.4.12 below. The quotients by $\Gamma(G_{\ell,P})$ of the bottom row fit into a tower:

$$(3.4.7.1) \qquad \overline{X}_{\ell,P} \to \overline{X}_{\ell,P}^{\mathrm{red}} \to X_{\ell,P}^{Sa} \to \widehat{\Sigma}_P',$$

though fixed points of $\Gamma(G_{\ell,P})$ at the boundary of the latter two prevents the $\Gamma(G_{\ell,P})$-quotient of (3.4.7) from being Cartesian.

We determine next:

Proposition 3.4.8. *The natural mappings associated to* (3.4.7),

$$(3.4.8.1) \quad \mathrm{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee) \to \mathrm{LCMb}(\overline{D}_{\ell,P}^{\mathrm{red}}, (\widehat{\Sigma}_P^c)^\vee) \to$$
$$\mathrm{LCMb}(D_{\ell,P}^{Sa}, (\widehat{\Sigma}_P^c)^\vee) \to \mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to (\widehat{\Sigma}_P^c)^\vee,$$

satisfy LCMb-*basechange* (1.1.8) *for* $(\widehat{\Sigma}_P^c)^\vee$. *Moreover, all are homotopy equivalences with contractible fibers, as are the projections onto* $(\widehat{\Sigma}_P^c)^\vee$. *The same holds for the mappings on* $\mathrm{LCMb}(\,\cdot\,, (\widehat{\Sigma}_P^c)^\vee)$ *induced by morphisms of Satake compactifications.*

Proof. We show that

$$(3.4.8.2) \qquad \partial \mathrm{LCM}(\overline{C}_P, (C_P)^{\mathrm{tor,exc}}) = \mathrm{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee) \simeq \widetilde{\mathcal{T}}^{BS},$$

the other cases being essentially the same. As in the proof of Proposition 3.3.8, one has the inclusion

$$(3.4.8.3) \qquad \mathrm{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee) \supseteq \widetilde{\mathcal{T}}^{BS}.$$

On the other hand, we have the tautology $(\widehat{\tau}^{BS} \times \tau^\vee) = \widehat{\tau}^{BS} \times_{\widehat{\Sigma}_P} (\widehat{\tau} \times \tau^\vee)$ for $\widehat{\tau} \in \widehat{\Sigma}_P$, and therefore

$$(3.4.8.4) \qquad \widetilde{\mathcal{T}}^{BS} \simeq \overline{D}_{\ell,P} \times_{\widehat{\Sigma}_P} \widetilde{\mathcal{T}} = \overline{D}_{\ell,P} \times_{\widehat{\Sigma}_P} \mathrm{LCMb}(\widehat{\Sigma}_P, (\widehat{\Sigma}_P^c)^\vee);$$

by (1.1.6),

$$(3.4.8.5) \qquad \mathrm{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee) \subseteq \overline{D}_{\ell,P} \times_{\widehat{\Sigma}_P} \mathrm{LCMb}(\widehat{\Sigma}_P, (\widehat{\Sigma}_P^c)^\vee).$$

We see from (3.4.8.3), (3.4.8.4) and (3.4.8.5) that (3.4.8.2) holds, and and then that the inclusion (3.4.8.5) is an equality. By parallel arguments, we get the same

for $\overline{D}_{\ell,P}^{\text{red}}$ and $D_{\ell,P}^{Sa}$. This yields LCMb-basechange for $(\widehat{\Sigma}_P^c)^\vee$ with respect to the mappings in the bottom row of (3.4.7).

The fibers in (3.4.8.1), over the interior of $\tau^\vee$ in $(\widehat{\Sigma}_P^c)^\vee$ are the respective closures $\widehat{\tau}^{BS}$, $\widehat{\tau}^{\text{red}}$, $\widehat{\tau}^{Sa}$ of $\widehat{\tau}^\circ$ in $\overline{D}_{\ell,P}$, $\overline{D}_{\ell,P}^{\text{red}}$ and $D_{\ell,P}^{Sa}$. These closures are contractible, and the mappings between them have contractible fibers, as can be seen from [Z3:(3.8)]; that also covers the case of a morphism of Satake compactifications. $\square$

(3.4.8.6) *Remark.* Lest it be forgotten, if one has contractible fibers for the LCMb, one has contractible fibers for the whole LCM, for trivial reasons.

We next deduce the associated result for $\Gamma(G_{\ell,P})$-quotients:

Proposition 3.4.9. *The natural mappings induced by* (3.4.7.1):

$$(3.4.9.1)\quad \text{LCMb}(\overline{X}_{\ell,P}, \widehat{\Sigma}_2'') \to \text{LCMb}(\overline{X}_{\ell,P}^{\text{red}}, \widehat{\Sigma}_2'') \to$$
$$\text{LCMb}(X_{\ell,P}^{Sa}, \widehat{\Sigma}_2'') \to \text{LCMb}(\widehat{\Sigma}_1', \widehat{\Sigma}_2'') \to \widehat{\Sigma}_2''$$

are all homotopy equivalences with contractible fibers. The same holds for the mappings on $\text{LCMb}(\,\cdot\,, \widehat{\Sigma}_2'')$ *induced by morphisms of Satake compactifications.*

Proof. Because $\Gamma(G_{\ell,P})$ acts freely on $(\widehat{\Sigma}_P^c)^\vee$, Corollary 3.4.2 would again apply. We see that again the issue in deducing our assertion from Proposition 3.4.8 is diagonality. The argument we present for $\overline{X}_{\ell,P}$ is easily seen to apply to the other cases.

Let $\widehat{\tau} \in \widehat{\Sigma}_1^\circ$, $e_1 \in \widehat{\tau}^{BS}$, and $e_2 \in (\tau^\vee)^\circ$. As $\widehat{\tau}$ varies, this gives all points (e_1, e_2) of $\text{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee)$, by the equality in (3.4.7.5). Suppose that $(\gamma \cdot e_1, e_2) \in \text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$ for some $\gamma \in \Gamma(G_{\ell,P})$. Let $\overline{e}_1$ be the projection of e_1 onto $\widehat{\Sigma}_1$. Then $\overline{e}_1 \in \widehat{\tau}$, $e_2 \in (\tau^\vee)^\circ$, and $(\gamma \cdot \overline{e}_1, e_2) \in \text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee)$. By (3.4.3.5, iii), $\gamma = 1$, so the actions are diagonal. This proves our assertion. $\square$

Having considered the mapping $\text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to (\widehat{\Sigma}_P^c)^\vee$ for Proposition 3.4.3, we look now at the other projection: $\text{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to \widehat{\Sigma}_1 \simeq \widehat{\Sigma}_P$. For $\widehat{\tau} \in \widehat{\Sigma}_P$, put

$$(3.4.10)\quad B(\widehat{\tau}) = \bigcup \{v^\vee : \widehat{v} \in \widehat{\Sigma}_P^\circ \text{ and } \widehat{\tau} \text{ is a face of } \widehat{v}\}$$
$$= \bigcup \{v^\vee : \widehat{v} \in \widehat{\Sigma}_P^\circ \text{ contains } \widehat{\tau} \text{ as a codimension-one face}\}.$$

We invoke the treatment of duals and joins from (3.3). The condition that $\widehat{v} \in \widehat{\Sigma}_P^\circ$ contains $\widehat{\tau}$ as a codimension one face can be rewritten as $\widehat{v} = \widehat{\tau} * \widehat{\alpha}$, where $\widehat{\alpha}$ is a

vertex of $\widehat{\Sigma}_P^c$. This allows us to write

$$(3.4.10.1) \qquad B(\widehat{\tau}) = \bigcup \{(\tau^\vee \cap \alpha^\vee) : \alpha \text{ as above}\} = \tau^\vee \cap (\Sigma_P^c)^\vee,$$

which provides another way of looking at (3.3.8.1). When $\widehat{\tau} \in \widehat{\Sigma}_P^\circ$, $B(\widehat{\tau})$ is just $\tau^\vee$; for $\widehat{\tau} \subset \widehat{\sigma}$, $B(\widehat{\tau}) \supseteq B(\widehat{\sigma})$. Also, one should not forget (3.2.5).

Lemma 3.4.11. *Let $\widehat{\tau} \in \widehat{\Sigma}_1$. Then $B(\widehat{\tau})$ is contractible.*

(3.4.11.1) *NB*—For general triangulated spaces, the set $B(\widehat{\tau})$ need not be contractible. Indeed, if $\widehat{\tau}$ were the vertex (declared to be "boundary") of the cone on a circle, $B(\widehat{\tau})$ would be a circle.

Proof of Lemma 3.4.11. This is easy if $\widehat{\tau} \in \widehat{\Sigma}_P^\circ$: $B(\widehat{\tau}) = \tau^\vee$, which is contractible by Lemma 3.2.4. If $\widehat{\tau}$ is in $\partial\widehat{\Sigma}_1$, $B(\widehat{\tau})$ is the boundary of $\tau^\vee$ in $(\widehat{\Sigma}_P^c)^\vee$ by (3.4.10.1). To see it is contractible, we will determine the link of the boundary component of $\widehat{C}_P$ that contains $\widehat{\tau}^\circ$.

Recall that the $\mathbb{Q}$-root system of $G_{\ell,P}$ is of type A, with simple roots of the form $\Delta_\ell := \{\beta_j : j < k\}$ inside the root system of type BC or C from (2.1). In particular, it too has a linear Dynkin diagram, and the picture is analogous to that of (2.1). The boundary components of $\widehat{C}_P$ are normalized by maximal $\mathbb{Q}$-parabolic subgroups $Q_{\ell,P}$ of $G_{\ell,P}$. The standard ones are determined by deleting a root from Δ_ℓ. The parameter (a non-empty set of simple roots) for this Satake compactification is $\{\beta_1\}$ (see [HZ2:§2.1]), on the opposite end of the Dynkin diagram for G from that for (2.1.4). The selection and omission of $\beta_j \in \Delta_\ell$ splits Δ_ℓ into two disjoint pieces of type A, which we write as $\Delta_\ell^- \sqcup \Delta_\ell^+$, with Δ_ℓ^- giving simple $\mathbb{Q}$-roots for the automorphism group, denoted $G_{\ell,P}^-$, of the boundary component D_ℓ^- of $\widehat{C}_P$. The corresponding Levi subgroup of $Q_{\ell,P}$ is an almost-direct product of the form $G_{\ell,P}^- \cdot G_{\ell,P}^+$, with the latter factor having Δ_ℓ^+ as simple $\mathbb{Q}$-roots.

We can now see that the link of D_ℓ^- is contractible. As allowed by the opening paragraph of section (2.3), we may take D there to be the non-Hermitian $D_{\ell,P}$. We can identify the link on $\overline{D}_{\ell,P}$. Thus, let $R = Q_{\ell,P}$ in (2.3.2). As $Q_{\ell,P}$ is maximal, we have that $\dim A_R = 1$. The link of D_ℓ^- is then given by an embedded copy of $D_\ell^+ \times W(\mathbb{R}) \subset D_{\ell,P}$, where the A_R-component in (2.3.2) is held constant; W is the unipotent radical of $Q_{\ell,P}$ and D_ℓ^+ is the symmetric space—taken in the sense of (2.1)—of $G_{\ell,P}^+$. The set $B(\widehat{\tau})$ is of the same homotopy type as the link, so it too contractible. $\square$

There are partial analogues of Propositions 3.4.8 and 3.4.9 "from the other side."

Proposition 3.4.12. i) *For $\widehat{\tau} \in \widehat{\Sigma}_1$, the fiber of*

(3.4.12.1) $$\mathrm{LCMb}(\widehat{\Sigma}_1, (\widehat{\Sigma}_P^c)^\vee) \to \widehat{\Sigma}_1$$

over $\widehat{\tau}^\circ$ is $B(\widehat{\tau})$. In particular,

(3.4.12.2) $\mathrm{LCMb}(\widehat{\Sigma}_1, (\Sigma_P^c)^\vee) = \bigsqcup \{\widehat{\tau}^\circ \times B(\widehat{\tau}) : \widehat{\tau} \in \widehat{\Sigma}_P\} = \bigcup \{\widehat{\tau} \times B(\widehat{\tau}) : \widehat{\tau} \in \widehat{\Sigma}_P\}$,

and the fibers in (3.4.12.1) *are contractible.*

ii) *It is analogous for the vertical mappings in* (3.4.7):
(3.4.12.3)
$\mathrm{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee) = \bigsqcup \{(\widehat{\tau}^{BS})^\circ \times B(\widehat{\tau}) : \widehat{\tau} \in \widehat{\Sigma}_P\} = \bigcup \{\widehat{\tau}^{BS} \times B(\widehat{\tau}) : \widehat{\tau} \in \widehat{\Sigma}_P\}$,

where $(\widehat{\tau}^{BS})^\circ$ denotes the inverse image in $\overline{D}_{\ell,P}$ of $\widehat{\tau}^\circ$; likewise for the other cases $(\overline{D}_{\ell,P}^{\mathrm{red}}$ and $D_{\ell,P}^{Sa})$.

(3.4.12.4) *Remark.* For $\tau \in \widehat{\Sigma}_1$, the mapping $(\widehat{\tau}^{BS})^\circ \to \widehat{\tau}^\circ$ is a homeomorphism if and only if $\widehat{\tau} \in \widehat{\Sigma}_P^\circ$. The same holds in the other cases.

Proof of Proposition 3.4.12. i) That the fiber of (3.4.12.1) over $\widehat{\tau}^\circ$ is $B(\widehat{\tau})$ is contained in (3.3.8.1). This implies the first equality in (3.4.12.2). The second equality follows from the fact that B is order-reversing on simplices, mentioned before Lemma 3.4.11. The fibers are contractible by Lemma 3.4.11.

ii) From (3.4.7), there is LCMb-basechange for $(\widehat{\Sigma}_P^c)^\vee$, so the spaces occurring as fibers in $\widetilde{\mathcal{T}}^{BS} \to \overline{D}_{\ell,P}$ are the same as those in $\widetilde{\mathcal{T}}^{Sa} \to D_{\ell,P}^{Sa}$. Taking $D_{\ell,P}^{Sa}$ to be $\widehat{\Sigma}_P$, we have from i) that (3.4.12.3) holds and the fibers are contractible. $\square$

Proposition 3.4.12 has an interesting consequence:

Corollary 3.4.13. *The mapping*

$$\mathrm{LCMb}(\overline{X}_{\ell,P}, \widehat{\Sigma}_2'') \to \overline{X}_{\ell,P}$$

has contractible fibers, so is a homotopy equivalence.

Proof. Our running hypothesis is that $\Gamma(G_{\ell,P})$ acts freely on $D_{\ell,P}$. The same holds for its action on $\overline{D}_{\ell,P}$ (a basic feature of the Borel-Serre construction; see [BS:9.5]). Moreover, the actions on $\overline{D}_{\ell,P}$ and $(\widehat{\Sigma}_P^c)^\vee$ are diagonal, as determined in the proof of Proposition 3.4.9. From Corollary 3.4.2, the fiber is determined prior to taking the $\Gamma(G_{\ell,P})$-quotient. From Proposition 3.4.12, ii), the fiber is $B(\widehat{\tau})$, which is contractible by Lemma 3.4.11. $\square$

(3.5) Fibers over the strata of X^*. We next proceed to show how Theorem 3.1.1 follows from a mild variant of Proposition 3.4.9. The assertions in (3.4) are all about boundaries attached to the homogeneous cones C_P, and these must be brought to bear upon D itself.

From §2, all of the compactifications of X under consideration admit a morphism onto X^*. We restrict our attention to the portion of these that maps to a neighborhood of the P-stratum of X^*, and then determine the LCM's. This is legitimate by Corollary 1.1.5.

For the Borel-Serre spaces, one starts with the decomposition associated to P:

$$(3.5.1) \qquad D = D_P \times D_{\ell,P} \times A_P \times W_P \qquad (A_P \simeq \mathbb{R}_{\geq 0}),$$

and adjoins accordingly (see (2.3)):

$$(3.5.2) \qquad \begin{array}{lll} \text{for } \overline{D} & : & D_P \times \overline{D}_{\ell,P} \times \{\infty\} \times W_P; \\ \text{for } \overline{D}^{\mathrm{exc}} & : & D_P \times \overline{D}_{\ell,P} \times \{\infty\} \times V_P; \\ \text{for } \overline{D}^{\mathrm{red}} & : & D_P \times \overline{D}_{\ell,P}^{\mathrm{red}} \times \{\infty\} \times \{1\}; \\ \text{for } D^* & : & D_P \times \{pt\} \times \{\infty\} \times \{1\}. \end{array}$$

We recall the basic determinations (see Propositions 2.3.8 and 2.3.10):

Proposition 3.5.3. *i) The fiber of $\overline{D} \to D^*$ over D_P is $\overline{D}_{\ell,P} \times W_P(\mathbb{R})$.*

ii) The fiber of $\overline{D}^{\mathrm{exc}} \to D^*$ over D_P is $\overline{D}_{\ell,P} \times V_P(\mathbb{R})$.

iii) The fiber of $\overline{D}^{\mathrm{red}} \to D^*$ over D_P is $\overline{D}_{\ell,P}^{\mathrm{red}}$. $\square$

We put $W_P' = \Gamma(W_P)\backslash W_P(\mathbb{R})$ and $V_P' = \Gamma(V_P)\backslash V_P(\mathbb{R})$. Passing to arithmetic quotients in Proposition 3.5.3, one gets:

Corollary 3.5.4. *i) The fiber of $\overline{X} \to X^*$ over M_P is a W_P'-fibration over $\overline{X}_{\ell,P}$.*

ii) The fiber of $\overline{X}^{\mathrm{exc}} \to X^*$ over M_P is a V_P'-fibration over $\overline{X}_{\ell,P}$.

iii) The fiber of $\overline{X}^{\mathrm{red}} \to X^*$ over M_P is $\overline{X}_{\ell,P}^{\mathrm{red}}$. $\square$

We will look at the above in relation to the fiber of $X^{\mathrm{tor,exc}} \to X^*$ over M_P. Recall from Proposition 3.1.2 that we can use (3.5.1) in the form $U_P(\mathbb{R})\backslash D \simeq C_P \times D_P \times V_P(\mathbb{R})$ for that purpose. The portion of the excentric toroidal boundary over M_P, like that of the excentric Borel-Serre, is adjoined by a construction on

$$A_P \times D_{\ell,P} \simeq C_P$$

(Again, one must not forget that the $U_P(\mathbb{R})$-quotient is actually taken only at the boundary, and not in the interior as the preceding may suggest.) The Baily-Borel-type P-stratum of the excentric toroidal boundary is an arithmetic quotient of

$\partial(U_P(\mathbb{R})^{\mathrm{im}})_{\Sigma_P^\circ} \times D_P \times V_P(\mathbb{R})$. It has a canonical mapping onto M_P (see (2.2.5.2), then take the (T_P^c)-quotient, as for (2.2.18)) that is induced by projection of that product onto D_P. Since $\partial U_P(\mathbb{R})_{\Sigma_P^\circ} \simeq (\widehat{\Sigma}_P^c)^\vee$, $\Gamma(G_{\ell,P})\backslash\partial U_P(\mathbb{R})_{\Sigma_P^\circ}$ coincides with $\widehat{\Sigma}_2''$ from (3.3), the toroidal analogue of Corollary 3.5.4 ii) takes the following form. Put $\Gamma' = \Gamma(G_{\ell,P} \cdot V_P)$.

Proposition 3.5.5. *i) The action of Γ' on $(\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R})$ is free.*

ii) The fiber of $X^{\mathrm{tor,exc}} \to X^*$ over M_P is $\Gamma'\backslash((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))$. It is a V_P'-fibration over $\widehat{\Sigma}_2''$. $\quad\square$

We can now proceed to determine the least common modifications. That entails adapting Proposition 3.4.9 to include the role of W_P. Let $\widetilde{\Gamma}' = \Gamma(G_{\ell,P} \cdot W_P)$.

Proposition 3.5.6. *i) The fibers over $M_P \subset X^*$ of the natural projections*

$$\mathrm{LCM}(\overline{X}, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}, \quad \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}, \quad \mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}$$

are given respectively by the rows of the commutative diagram:

$$
(3.5.6.1) \quad
\begin{array}{ccc}
\mathrm{LCMb}(\widetilde{\Gamma}'\backslash(\overline{D}_{\ell,P} \times W_P(\mathbb{R})), \Gamma'\backslash((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) & \longrightarrow & \Gamma'\backslash(\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}) \\
\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{LCMb}(\Gamma'\backslash(\overline{D}_{\ell,P} \times V_P(\mathbb{R})), \Gamma'\backslash((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) & \longrightarrow & \Gamma'\backslash((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R})) \\
\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{LCMb}(\overline{X}_{\ell,P}^{\mathrm{red}}, \Gamma'\backslash((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) & \longrightarrow & \Gamma'\backslash((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))
\end{array}
$$

ii) The rows in (3.5.6.1) are surjective and have contractible fibers.

Proof. The process is familiar by now: we consider the situation before arithmetic quotients are taken, viz.,

$$
(3.5.6.2) \quad
\begin{array}{ccc}
\mathrm{LCMb}((\overline{D}_{\ell,P} \times W_P(\mathbb{R})), ((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) & \longrightarrow & (\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}) \\
\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{LCMb}((\overline{D}_{\ell,P} \times V_P(\mathbb{R})), ((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) & \longrightarrow & (\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}) \\
\downarrow & & \downarrow{\scriptstyle =} \\
\mathrm{LCMb}(\overline{D}_{\ell,P}^{\mathrm{red}} \times \{1\}, ((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) & \longrightarrow & ((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))
\end{array}
$$

and then verify diagonality for ii).

To do the analogue of i) over D_P, we may set aside the common factor of D_P in (3.5.2), and view the issue as one of determining the corresponding LCM's and

projections in $\mathfrak{PCp}(D_{\ell,P} \times A_P \times W_P(\mathbb{R}))$. These are given in LCMb format in (3.5.6.2). We have the following inputs:

$$\mathrm{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee) \simeq \widetilde{\mathcal{T}}^{BS} \quad \text{and} \quad \mathrm{LCMb}(\overline{D}_{\ell,P}^{\mathrm{red}}, (\widehat{\Sigma}_P^c)^\vee) \simeq \widetilde{\mathcal{T}}^{\mathrm{red}}$$

from (3.4.8.1); morphisms in $\mathfrak{PCp}(A_P \times W_P(\mathbb{R}))$ imply: $\mathrm{LCMb}(W_P(\mathbb{R}), V_P(\mathbb{R})) \simeq W_P(\mathbb{R})$, $\mathrm{LCMb}(V_P(\mathbb{R}), V_P(\mathbb{R})) \simeq V_P(\mathbb{R})$, and $\mathrm{LCMb}(\{1\}, V_P(\mathbb{R})) \simeq V_P(\mathbb{R})$. Using Lemma 1.1.2, we see we may take products and combine these to obtain:

$$(3.5.6.3) \qquad \mathrm{LCMb}((\overline{D}_{\ell,P} \times W_P(\mathbb{R})), ((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) \simeq \widetilde{\mathcal{T}}^{BS} \times W_P(\mathbb{R});$$

$$(3.5.6.4) \qquad \mathrm{LCMb}((\overline{D}_{\ell,P} \times V_P(\mathbb{R})), ((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) \simeq \widetilde{\mathcal{T}}^{BS} \times V_P(\mathbb{R});$$

$$(3.5.6.5) \qquad \mathrm{LCMb}(\overline{D}_{\ell,P}^{\mathrm{red}} \times \{1\}, ((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) \simeq \widetilde{\mathcal{T}}^{\mathrm{red}} \times V_P(\mathbb{R}).$$

With $W_P(\mathbb{R})$ and $V_P(\mathbb{R})$ being contractible, the horizontal morphisms in (3.5.6.2) are, up to homotopy, given by the first two in Proposition 3.4.8. In particular, the fibers in (3.5.6.2) are contractible.

However, we must pass to the arithmetic quotients for (3.5.6.1). For that we can appeal to the diagonality given in Proposition 1.3.7. Because of Proposition 3.5.5, i), Corollary 3.4.2 applies in determining the fibers; they are the same as in (3.5.6.2). With the aid of Proposition 3.4.3, we obtain from (3.5.6.3) that the fiber in the first row, over the point represented by $(q, w) \in (\widehat{\Sigma}_P^c)^\vee \times W_P(\mathbb{R})$, is $\widehat{\tau}^{BS} \times [w + U_P]$ whenever $q \in (\tau^\vee)^\circ$. Similarly, from (3.5.6.4) in the second row the fiber is $\widehat{\tau}^{BS} \times [v]$, and from (3.5.6.5) in the third row of (3.5.6.1) the fiber is $\widehat{\tau}^{\mathrm{red}}$. (Here, $\widehat{\tau}^{BS}$ and $\widehat{\tau}^{\mathrm{red}}$ are as in (3.4.5).) In all three cases, the fiber is contractible. $\square$

(3.5.6.6) *Remark.* Over the interior of $\tau^\vee$, the left-hand column in (3.5.6.1) is given by the canonical projections

$$\widehat{\tau}^{BS} \times \{w\} \to \widehat{\tau}^{BS} \times \{v\} \to \widehat{\tau}^{\mathrm{red}} \qquad (v = w + U_P).$$

Corollary 3.5.7. *For Y a Borel-Serre space (i.e., $Y = \overline{X}$, $Y = \overline{X}^{\mathrm{exc}}$, and $Y = \overline{X}^{\mathrm{red}}$), the mapping*

$$\mathrm{LCM}(Y, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}$$

has contractible fibers, so is a homotopy equivalence.

Proof. Over $M_P \subset \partial X^*$, the fiber of $\mathrm{LCM}(Y, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}$ is given by that of the corresponding row of (3.5.6.1). We have already seen that this is contractible for all P (Proposition 3.5.6). We invoke the criterion from [GT,§8]: a morphism of compact stratified spaces having contractible fibers is a homotopy equivalence. $\square$

By inverting the above homotopy equivalence in the second case, we obtain a homotopy class of mappings (analogous to $h : X^{\mathrm{tor}} \to \overline{X}^{\mathrm{red}}$ in [GT]):

$$(3.5.8) \qquad\qquad k : X^{\mathrm{tor,exc}} \to \overline{X}^{\mathrm{exc}}.$$

We can actually take this further by considering the other projection,

$$(3.5.9) \qquad\qquad \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) \to \overline{X}^{\mathrm{exc}}$$

Theorem 3.5.10. *The canonical mapping* (3.5.9) *has contractible fibers, so is a homotopy equivalence.*

Proof. The proof follows the same lines as that of Corollary 3.5.7. Over a point of $M_P \subset \partial X^*$, the fiber of $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) \to \overline{X}^{\mathrm{exc}}$ is given by

$$\mathrm{LCMb}(\Gamma'\backslash(\overline{D}_{\ell,P} \times V_P(\mathbb{R})), \Gamma'\backslash((\widehat{\Sigma}_P^c)^\vee \times V_P(\mathbb{R}))) \to \Gamma'\backslash(\overline{D}_{\ell,P} \times V_P(\mathbb{R})).$$

Because Γ' acts diagonally on (3.5.6.4), and freely on $\overline{D}_{\ell,P} \times V_P(\mathbb{R})$, we again apply Corollary 3.4.2 here to determine the fiber. Thus, we may forget about Γ', and then drop the common factor of $V_P(\mathbb{R})$. This reduces us to considering the fibers of $\mathrm{LCMb}(\overline{D}_{\ell,P}, (\widehat{\Sigma}_P^c)^\vee) \to \overline{D}_{\ell,P}$ for all P. This was already treated in Corollary 3.4.13: the fibers are contractible. $\square$

Putting Corollary 3.5.7 and Theorem 3.5.10 together, we obtain the following fundamental assertion, which is cited in [Z5:§11].

Corollary 3.5.11. *The spaces* $\overline{X}^{\mathrm{exc}}$ *and* $X^{\mathrm{tor,exc}}$ *are homotopy equivalent.* $\square$

We finish this section with a further LCM-basechange result.

Proposition 3.5.12. LCM-*basechange holds for* $\overline{X}^{\mathrm{red}}$ *and* $\overline{X}^{\mathrm{exc}}$ *with respect to the morphism* $X^{\mathrm{tor}} \to X^{\mathrm{tor,exc}}$.

Proof. To prove that the assertion holds at the boundary, we must show that for all maximal parabolic P,

(3.5.12.1)
$$\mathrm{LCMb}(\partial_P \overline{X}^{\mathrm{exc}}, \partial_P X^{\mathrm{tor}}) = \mathrm{LCMb}(\partial_P \overline{X}^{\mathrm{exc}}, \partial_P X^{\mathrm{tor,exc}}) \times_{\partial_P X^{\mathrm{tor,exc}}} \partial_P X^{\mathrm{tor}},$$

where ∂_P indicates the part of the boundaries mapping to $M_P \subset \partial X^*$ (i.e., LCMb-basechange holds there), and the same with $\overline{X}^{\mathrm{exc}}$ replaced by $\overline{X}^{\mathrm{red}}$. We are in the LCMb-basechange variant of a situation from §1. The torus T_P^c acts trivially on both $\partial_P \overline{X}^{\mathrm{exc}}$ and $\partial_P \overline{X}^{\mathrm{red}}$, and $\partial_P X^{\mathrm{tor,exc}}$ is the T_P^c-quotient of $\partial_P X^{\mathrm{tor}}$. The argument proving (3.5.12.1) follows the one in the proof of Proposition 1.1.14 verbatim. $\square$

From this, we obtain the following:

Corollary 3.5.13. *i) The canonical mapping* $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor}}) \to X^{\mathrm{tor}}$ *has contractible fibers.*

ii) [GT] *The canonical mapping* $\mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) \to X^{\mathrm{tor}}$ *has contractible fibers.*

Proof. From the definition (1.1.8), the LCM-basechange given by Proposition 3.5.12 implies that $\mathrm{LCM}(Y, X^{\mathrm{tor}}) \to X^{\mathrm{tor}}$ and $\mathrm{LCM}(Y, X^{\mathrm{tor,exc}}) \to X^{\mathrm{tor,exc}}$ have the same fiber when either $Y = \overline{X}^{\mathrm{red}}$ or $Y = \overline{X}^{\mathrm{exc}}$. These fibers are contractible by Corollary 3.5.7, so we are done. $\square$

4. Canonical extension of homogeneous vector bundles.

In this section, we perform the toroidal construction of [AMRT] (see our (2.2)) on a homogeneous vector bundle $\mathcal{E}_\Gamma$ on $X = \Gamma \backslash D$ (we drop henceforth the subscript "Γ"). This yields a holomorphic vector bundle $\mathcal{E}^{\mathrm{tor}}$ on X^{tor}. We show (4.4.5) that $\mathcal{E}^{\mathrm{tor}}$ is the canonical extension of $\mathcal{E}$ to X^{tor} in the sense of [Mu] (see [HZ1: 3.2]) when $\mathcal{E}$ is holomorphic. It is then an easy matter to descend $\mathcal{E}^{\mathrm{tor}}$ to a complex vector bundle $\mathcal{E}^{\mathrm{tor,exc}}$ on $X^{\mathrm{tor,exc}}$. The treatment is similar in tone to the Borel-Serre construction for the canonical extension $\overline{\mathcal{E}}^{\mathrm{red}}$ of $\mathcal{E}$ to $\overline{X}^{\mathrm{red}}$ given in [Z4: 1.10]; the latter also gives immediately the canonical extension bundle $\overline{\mathcal{E}}^{\mathrm{exc}}$ on $\overline{X}^{\mathrm{exc}}$.

(4.1) Standard notions. Let $D = G/K$ as before. Let E be a finite-dimensional complex vector space, and $\rho : K \to \mathrm{GL}(E)$ a representation of K. The action of K on E extends to one of its complexification $K(\mathbb{C})$, and thereby to the (correct choice of) $\mathbb{C}$-parabolic subgroup $\mathfrak{P}$ having $K(\mathbb{C})$ as Levi quotient. The so-called *compact dual* of D is given as $\check{D} = G(\mathbb{C})/\mathfrak{P}$, and it contains D as an open subset. The G-homogeneous vector bundle $\widetilde{\mathcal{E}} = G \times^K E$ on D descends to $\mathcal{E}$ on X by taking the quotient by Γ on the left. It also extends to the $G(\mathbb{C})$-homogeneous vector bundle $\check{\mathcal{E}} = G(\mathbb{C}) \times^{\mathfrak{P}} E$ on $\check{D}$, with the bundle projection given by

$$(4.1.1) \qquad \check{\mathcal{E}} = G(\mathbb{C}) \times^{\mathfrak{P}} E \to G(\mathbb{C}) \times^{\mathfrak{P}} \{0\} = \check{D}.$$

(4.1.2) *Remark.* Taking $E = \{0\}$ gives $\widetilde{\mathcal{E}} = D$ and $\check{\mathcal{E}} = \check{D}$. Thus the constructions of [AMRT] will become a special case of ours. For this reason, we shall cease to talk about D and $\check{D}$ unless that is needed to clarify the discussion for the general homogeneous vector bundle.

Let P be a maximal parabolic subgroup of G. This determines the open subset $\check{D}(P) = U_P(\mathbb{C}) \cdot D$ of $\check{D}$ (cf. (3.1.2.2)). It is convenient to allow the improper parabolic G here, and then $P \prec G$ for all maximal parabolic P and $\check{D}(G) = D$. One sees that $\check{D}(P) \subset \check{D}(Q)$ whenever $Q \prec P$, as $U_Q \supset U_P$. We point out that the complement of $\check{D}(P)$ in $\check{D}(Q)$ has non-empty interior.

Moving K by the inverse Cayley transform for P (see [HZ1: 1.8]) determines a basepoint for $\check{D}(P)$ that is left fixed by $G_{\ell,P}$. One sees that $\check{D}(P)$ is homogeneous under the group $P' = G_{h,P} \cdot W_P \cdot U_P(\mathbb{C})$. As $P' \cap K = K_{h,P}$, where $K_{h,P} = K \cap G_{h,P}$, one obtains the decomposition

$$(4.1.3) \qquad \check{D}(P) \simeq D_P \times V_P \times U_P(\mathbb{C})$$

(see (3.1.2.2); also (2.2.2)). The action of $G_{\ell,P}$ on $\check{D}(P)$ is induced by its adjoint action on P. It preserves the factors in (4.1.3); in particular, itsufficiently small) is trivial on the factor D_P. This yields a projection of $D \subset \check{D}(P)$ onto D_P.

Let $\check{\mathcal{E}}(P)$ denote the restriction of $\check{\mathcal{E}}$ to $\check{D}(P)$. We can write the bundle projection, the restriction of (4.1.1), as

$$(4.1.4) \qquad \check{\mathcal{E}}(P) = P' \times^{K_{h,P}} E \to P' \times^{K_{h,P}} \{0\} = \check{D}(P),$$

with P' acting on the left. We have $\check{\mathcal{E}}(P) \subset \check{\mathcal{E}}(Q)$ whenever $Q \prec P$.

(4.2) Torus actions and torus embeddings, revisited. We form the quotient $\check{\mathcal{E}}'_P = \Gamma(P') \backslash \check{\mathcal{E}}(P)$. On $P' \times E$, the actions of $K_{h,P}$ (as for (4.1.4)) and the $\mathbb{C}$-torus $T_P = \Gamma(U_P) \backslash U_P(\mathbb{C})$ (trivial on E) commute, as U_P is the center of P'. Therefore, T_P acts on $\check{\mathcal{E}}(P)'$. We get a commutative diagram:

$$(4.2.1)$$
$$
\begin{array}{ccc}
\check{\mathcal{E}}'_P & \xrightarrow{\check{\pi}_2} & \check{\mathcal{E}}^A_P \\
\downarrow & & \downarrow{\scriptstyle \pi_A} \\
M'_P & \xrightarrow{\pi_2} & \mathcal{A}_P
\end{array}
$$

with the rows giving the T_P-quotients; it is the $\Gamma(P')$-quotient of

$$(4.2.1.1)$$
$$
\begin{array}{ccc}
P' \times^{K_{h,P}} E & \longrightarrow & (P'/T_P) \times^{K_{h,P}} E \\
\downarrow & & \downarrow \\
P' \times^{K_{h,P}} \{0\} & \longrightarrow & (P'/T_P) \times^{K_{h,P}} \{0\}.
\end{array}
$$

For any fan Σ_P in U_P, let T_{P,Σ_P} be the corresponding torus embedding, as in (2.2). Put

(4.2.2) $$\check{\mathcal{E}}'_{P,\Sigma_P} = \check{\mathcal{E}}'_P \times^{T_P} T_{P,\Sigma_P} .$$

When $E = 0$, this is the basic building block of the toroidal construction in [AMRT] (see our (2.2.5.1)), and it plays the same role here. Moreover, the toroidal construction (4.2.2) yields

(4.2.3) $$\check{\pi}_{2,\Sigma_P} : \check{\mathcal{E}}'_{P,\Sigma_P} = \check{\mathcal{E}}'_P \times^{T_P} T_{P,\Sigma_P} \to \check{\mathcal{E}}^A_P ,$$

and this fits into a commutative diagram

(4.2.4)
$$\begin{array}{ccc}
\check{\mathcal{E}}'_{P,\Sigma_P} & \xrightarrow{\check{\pi}_{2,\Sigma_P}} & \check{\mathcal{E}}^A_P \\
\downarrow & & \downarrow{\scriptstyle \pi_A} \\
M'_{P,\Sigma_P} & \xrightarrow{\pi_{2,\Sigma_P}} & \mathcal{A}_P
\end{array}$$

that extends (4.2.1). In terms of (4.2.1.1), this is

(4.2.4.1)
$$\begin{array}{ccc}
P' \times^{K_{h,P}} E \times^{T_P} T_{P,\Sigma_P} & \longrightarrow & P' \times^{K_{h,P}} E \times^{T_P} \{pt\} \\
\downarrow & & \downarrow \\
P' \times^{K_{h,P}} \{0\} \times^{T_P} T_{P,\Sigma_P} & \longrightarrow & P' \times^{K_{h,P}} \{0\} \times^{T_P} \{pt\}
\end{array}$$

In particular, (4.2.3) is a T_{P,Σ_P}-fibration. We will show that (4.2.4), so also (4.2.1), is Cartesian in Proposition 4.3.2 below.

Finally, we use the hypothesis that Σ_P is $\Gamma(G_{\ell,P})$-equivariant. Then $\Gamma(G_{\ell,P})$ acts on $\check{\mathcal{E}}'_{P,\Sigma_P}$: for $\ell \in \Gamma(G_{\ell,P})$, $p' \in P'$, and $t \in T_{P,\Sigma_P}$, $\ell \cdot (p', e, t) = (\ell p' \ell^{-1}, \ell \cdot e, \ell \cdot t)$ in terms of (4.2.4.1). We check this is compatible with the actions of $k \in K_{h,P}$ and $s \in T_P$:

$$\begin{aligned}
\ell \cdot (p' s^{-1} k^{-1}, ke, st) &= (\ell p' s^{-1} k^{-1} \ell^{-1}, \ell k \cdot e, \ell \cdot st) \\
&= ((\ell p' \ell^{-1}(\ell s^{-1} \ell^{-1}) k^{-1}, k\ell \cdot e, (\ell s \ell^{-1}) \ell \cdot t) \\
&\sim (\ell p' \ell^{-1}, \ell \cdot e, \ell \cdot t)
\end{aligned}$$

Therefore, we can form $\Gamma(G_{\ell,P}) \backslash \check{\mathcal{E}}'_{P,\Sigma_P}$, a partial compactification of $\Gamma(P) \backslash \check{\mathcal{E}}(P)$.

(4.3) Identification of canonical extensions. For a topological vector bundle $\mathcal{V}$, let $\mathcal{C}(\mathcal{V})$ denote the sheaf of continuous sections of $\mathcal{V}$; if $\mathcal{V}$ is holomorphic, we denote by $\mathcal{O}(\mathcal{V})$ the sheaf of holomorphic sections. We begin by stating some basic facts about the unextended bundles:

Lemma 4.3.1. *i) The $G_{\ell,P}$-equivariant mapping $\check{\mathcal{E}}'_P \to M'_P$ in (4.2.1) induces the mapping $\check{\mathcal{E}}^A_P \to \mathcal{A}_P$ (by T_P-quotient), which is a vector bundle projection.*

ii) $\check{\mathcal{E}}'_P \simeq \pi^*_2 \check{\mathcal{E}}^A_P = M'_P \times_{\mathcal{A}_P} \check{\mathcal{E}}^A_P$.

iii) In terms of ii), the T_P-action on $\check{\mathcal{E}}'_P$ is given by the T_P-action on M'_P and the trivial action on $\check{\mathcal{E}}^A_P$. In particular, $[(\pi_2)_* \mathcal{C}(\check{\mathcal{E}}'_P)]^{T_P} = \mathcal{C}(\check{\mathcal{E}}^A_P)$.

Proof. Statements i) and ii) are immediate from (4.2.1.1). Note that we can then write $\check{\pi}_2$ as

$$M'_P \times_{\mathcal{A}_P} \check{\mathcal{E}}^A_P \to \mathcal{A}_P \times_{\mathcal{A}_P} \check{\mathcal{E}}^A_P;$$

iii) is now evident. $\quad\square$

We continue by giving next the extension of Lemma 4.3.1 over M'_{P,Σ_P}. Let $\pi_{2,\Sigma_P} : M'_{P,\Sigma_P} \to \mathcal{A}_P$ be as in (4.2.4).

Proposition 4.3.2. *i) Diagram (4.2.4) is a pullback diagram. In particular, the mapping $\check{\mathcal{E}}'_{P,\Sigma_P} \to M'_{P,\Sigma_P}$ is a vector bundle projection, and $\check{\mathcal{E}}'_{P,\Sigma_P} \simeq \pi^*_{2,\Sigma_P} \check{\mathcal{E}}^A_P$.*

ii) Moreover, $[(\pi_2)_* \mathcal{C}(\check{\mathcal{E}}'_{P,\Sigma_P})]^{T_P} = \mathcal{C}(\check{\mathcal{E}}^A_P)$.

Proof. Let O be an open subset of $\mathcal{A}_P$ over which both fibrations $\check{\pi}_2$ and b in (4.2.1) are trivial. We can then trivialize (4.2.1.1) over O as

(4.3.2.1)
$$
\begin{array}{ccc}
E \times O \times T_P & \xrightarrow{\check{\pi}_2} & E \times O \\
\downarrow & & \downarrow{\scriptstyle \pi_A} \\
O \times T_P & \xrightarrow{\pi_2} & O
\end{array}
$$

This trivialization extends when we replace T_P by T_{P,Σ_P}. Thus, the diagram (4.2.4) is Cartesian. All of the assertions follow. $\quad\square$

(4.3.2.2) *Remark.* As T_P acts trivially on $\check{\mathcal{E}}^A_P$, we can view Proposition 4.3.2 i) as

$$\check{\mathcal{E}}'_{P,\Sigma_P} = (M'_P \times_{\mathcal{A}_P} \check{\mathcal{E}}^A_P) \times^{T_P} T_{P,\Sigma_P} = (M'_P \times^{T_P} T_{P,\Sigma_P}) \times_{\mathcal{A}_P} \check{\mathcal{E}}^A_P = \pi^*_{2,\Sigma_P} \check{\mathcal{E}}^A_P.$$

Corollary 4.3.3. *The canonical extension of $\check{\mathcal{E}}'_P$, from M'_P to M'_{P,Σ_P}, is $\check{\mathcal{E}}'_{P,\Sigma_P}$.*

Proof. By definition (see [HZ1, 3.2]), the canonical extension of $\check{\mathcal{E}}'_P$ is $\pi^*_{2,\Sigma_P} \check{\mathcal{E}}^A_P$, in view of Lemma 4.3.1 ii). This coincides with $\check{\mathcal{E}}'_{P,\Sigma_P}$ by Proposition 4.3.2 i). $\quad\square$

To see that, as P varies, the $\check{\mathcal{E}}'_{P,\Sigma_P}$'s can be glued to produce a vector bundle on X^{tor}, we start by following [AMRT: III, §5]. Suppose that $Q \prec P$, so $\check{\mathcal{E}}(P) \subset \check{\mathcal{E}}(Q)$

(the complement has non-empty interior, for the same holds in $\check{D}(Q)$). One can decompose T_Q as the product of T_P and a complementary torus—call it $T_{P,Q}$. Assuming that $\Sigma_P \subseteq U_P \cap \Sigma_Q$, we get that

(4.3.4) $\check{\mathcal{E}}(Q)'_{\Sigma_P} = \check{\mathcal{E}}(Q)' \times^{T_Q} T_{Q,\Sigma_P} = \check{\mathcal{E}}(Q)' \times^{T_Q} (T_{P,Q} \times T_{P,\Sigma_P}) \simeq \check{\mathcal{E}}(Q)' \times^{T_P} T_{P,\Sigma_P}.$

From this, we see that $\check{\mathcal{E}}(Q)'_{\Sigma_P}$ contains the $\Gamma(T_{P,Q})$-quotient of $\check{\mathcal{E}}(P)'_{\Sigma_P}$ as an open set. Since $G_{\ell,P} \subset G_{\ell,Q}$, we obtain that $\Gamma(G_{\ell,P}) \backslash \check{\mathcal{E}}'_{P,\Sigma_P} \to \Gamma(G_{\ell,Q}) \backslash \check{\mathcal{E}}'_{Q,\Sigma_P}$ is a covering space over its image. As $\check{\mathcal{E}}'_{Q,\Sigma_P}$ is a dense open subset of $\check{\mathcal{E}}'_{Q,\Sigma_Q}$:

Proposition 4.3.5. $\Gamma(G_{\ell,P}) \backslash \check{\mathcal{E}}'_{P,\Sigma_P}$ *is a covering space over an open subset of* $\Gamma(G_{\ell,Q}) \backslash \check{\mathcal{E}}'_{Q,\Sigma_Q}$. $\square$

It is time to impose the usual *compatibility conditions* on the collection of fans $\Sigma = \{\Sigma_P\}$: $\Sigma_P = \Sigma_Q \cap U_P$ whenever $Q \prec P$, and $\Sigma_{\mathrm{Int}(\gamma)P} = \mathrm{Int}(\gamma)\Sigma_P$ for all $\gamma \in \Gamma$. We present next the construction of $\mathcal{E}^{\mathrm{tor}}$ (when $E = 0$, it reduces to the main theorem of [AMRT, III,§§5,6], treated in our (2.2)).

Only to put the construction of the vector bundle in "usual" patching format, we choose, for each P (allowing, for convenience, $S_G = D$), a *realm of reduction* $S_P \subset D$ *for* P. By this, we mean that S_P is a sufficiently large $\Gamma(P)$-invariant open set on which Γ-equivalence reduces to $\Gamma(P)$-equivalence. If we select S_P in a manner that is compatible with the action of Γ by conjugation on the set of parabolic subgroups, we need make only finitely many arbitrary choices. For instance, we can take S_P to be the inverse image in D of a suitable deleted collar of the closed Borel-Serre face $\overline{e'(P)} \subset \overline{X}$. Let $\mathcal{B}_P = \mathcal{E}(S_P)$ (the restriction of $\widetilde{\mathcal{E}}$ to S_P). We define $\mathcal{B}'_{P,\Sigma}$ to be the interior of the closure of the image of $\mathcal{B}_P$ in $\Gamma(G_{\ell,P}) \backslash \check{\mathcal{E}}'_{P,\Sigma_P}$. In particular, $\widetilde{\mathcal{E}} = \mathcal{E}_{G,\Sigma_G}$, with $\mathcal{B}_G = \widetilde{\mathcal{E}}$, $G_{\ell,G} = G$, and $\Sigma_G = \{0\}$. Let $S'_{P,\Sigma}$ denote the same for the zero vector bundle, i.e., for X itself, so

(4.3.6) $$\bigcup_P S'_{P,\Sigma_P} = X^{\mathrm{tor}}.$$

Proposition 4.3.7. *With identifications induced from Proposition* 4.3.5, $\bigcup_P \mathcal{B}'_{P,\Sigma_P}$ *is a vector bundle over* $X^{\mathrm{tor}} = \bigcup_P S'_{P,\Sigma_P}$, *which we denote* $\mathcal{E}^{\mathrm{tor}}$.

Proof. We have, from Lemma 4.3.1 i) by restriction, that each $\mathcal{B}'_{P,\Sigma_P}$ is a vector bundle over S'_{P,Σ_P}. The restrictions of Proposition 4.3.2 and 4.3.5 imply that they patch to define a vector bundle over X^{tor}. $\square$

Since the characterization of canonical extension can be given in terms of the homogeneous bundle on the sets M'_{P,Σ_P} (for all P) [HZ1: 3.2], we obtain immediately from Proposition 4.3.3:

Proposition 4.3.8. $\mathcal{E}^{\mathrm{tor}}$ *is the canonical extension (in the sense of* [Mu]*) of* $\mathcal{E}$ *from* X *to* X^{tor}. $\square$

We now define a (topological) vector bundle $\mathcal{E}^{\mathrm{tor,exc}}$ on $X^{\mathrm{tor,exc}}$ such that $\mathcal{E}^{\mathrm{tor}}$ is the pullback of $\mathcal{E}^{\mathrm{tor,exc}}$ under the quotient mapping $q : X^{\mathrm{tor}} \to X^{\mathrm{tor,exc}}$ (from (2.2.18)). We use the stratification of $X^{\mathrm{tor}} \to X^{\mathrm{tor,exc}}$ to do the same for $\mathcal{E}^{\mathrm{tor}}$. Note that for P maximal ${}^{<}Z_P \subset S'_{P,\Sigma_P}$, and it is for ${}^{<}Z_P$ that one takes the T^c_P-quotient to obtain the stratum ${}^{<}Z^{\mathrm{exc}}_P$ of $X^{\mathrm{tor,exc}}$. For convenience, we write for $P = G$, ${}^{<}Z_G = X$, and use it for the interior in:

$$(4.3.9.1) \qquad X^{\mathrm{tor}} = \bigsqcup_P {}^{<}Z_P \quad \text{and} \quad X^{\mathrm{tor,exc}} = \bigsqcup_P {}^{<}Z^{\mathrm{exc}}_P.$$

We can also write

$$(4.3.9.2) \qquad \mathcal{E}^{\mathrm{tor}} = \bigsqcup_P \mathcal{E}^{\mathrm{tor}}({}^{<}Z_P)$$

though these do not give *open* covers. We then put accordingly

$$(4.3.9.3) \qquad \mathcal{E}^{\mathrm{tor,exc}} = \bigsqcup_P \mathcal{E}^{\mathrm{tor,exc}}({}^{<}Z^{\mathrm{exc}}_P) \to X^{\mathrm{tor,exc}},$$

where $\mathcal{E}^{\mathrm{tor,exc}}({}^{<}Z^{\mathrm{exc}}_P) = \mathcal{E}^{\mathrm{tor}}({}^{<}Z_P)/T^c_P$, as a quotient of $\mathcal{E}^{\mathrm{tor}} \to X^{\mathrm{tor}}$. Because the T_P action is given as in Lemma 4.3.1 (iii), we see from (4.3.2.1) that

Proposition 4.3.10. $\mathcal{E}^{\mathrm{tor,exc}}$ *is a vector bundle (i.e., locally trivial) over the space* $X^{\mathrm{tor,exc}}$*, with* $q^*\mathcal{E}^{\mathrm{tor,exc}} \simeq \mathcal{E}^{\mathrm{tor}}$. $\square$

(4.4) The Goresky-Tai conjecture. We recall the statement of the conjecture (also given in [Z4: p.954]):

Conjecture A [GT: 9.5]**.** *Let* $h : X^{\mathrm{tor}} \to \overline{X}^{\mathrm{red}}$ *be any of the continuous mappings constructed in* [GT]*. Then the canonical extension* $\mathcal{E}^{\mathrm{tor}}$ *is topologically isomorphic to the pullback* $h^*\overline{\mathcal{E}}^{\mathrm{red}}$.

We have been leading up to the following rather natural result:

Theorem 4.4.1. *Suppose that for a mapping* $k : X^{\mathrm{tor,exc}} \to \overline{X}^{\mathrm{exc}}$ *as in* (3.5.12)*,* $\mathcal{E}^{\mathrm{tor,exc}} \simeq k^*\overline{\mathcal{E}}^{\mathrm{exc}}$. *Then* Conjecture A *is true.*

Proof. Since pullbacks of vector bundles are, up to isomorphism, determined by the homotopy class of the morphism, it is enough to show that there exist h and k in the homotopy classes for which our hypothesis implies Conjecture A.

We have the following picture:

$$
\begin{array}{ccccccc}
\mathcal{E}^{\mathrm{tor,exc}} & & & & & & \overline{\mathcal{E}}^{\mathrm{exc}} \\
\downarrow & & & & & & \downarrow \\
X^{\mathrm{tor,exc}} & \xleftarrow{\;\bullet\;} & \mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor,exc}}) & \xleftarrow{\;\bullet\;} & \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) & \xrightarrow[k]{\;\bullet\;} & \overline{X}^{\mathrm{exc}} \\
\uparrow {\scriptstyle q} & & \uparrow {\scriptstyle *} & & \uparrow {\scriptstyle *} & & \downarrow {\scriptstyle \overline{q}} \\
X^{\mathrm{tor}} & \xleftarrow[h]{\;\bullet\;} & \mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) & \xleftarrow{\;\bullet\;} & \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor}}) & \rightarrow & \overline{X}^{\mathrm{red}} \\
\uparrow & & & & & & \uparrow \\
\mathcal{E}^{\mathrm{tor}} & & & & & & \overline{\mathcal{E}}^{\mathrm{red}}
\end{array}
$$

where (we appeal to Corollaries (3.5.7) and (3.5.13)) an arrow labelled with a dot has a homotopy inverse and one labelled with an asterisk is an LCM-basechange. When the homotopy inverses are taken, the composite in the upper row gives k and that of the lower row gives h. We need to check that we can make choices so that the diagram commutes, for then $\overline{q}kq = h$ and

$$
h^*\overline{\mathcal{E}}^{\mathrm{red}} = q^*k^*\overline{q}^*\overline{\mathcal{E}}^{\mathrm{red}} = q^*k^*\overline{\mathcal{E}}^{\mathrm{exc}} = q^*\mathcal{E}^{\mathrm{tor,exc}} = \mathcal{E}^{\mathrm{tor}}.
$$

For that, we can take homotopy inverses for both marked arrows in the first row. This implies (by LCM-basechange) the same for the second row, such that

$$
\begin{array}{ccccc}
X^{\mathrm{tor,exc}} & \longrightarrow & \mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor,exc}}) & \longrightarrow & \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}}) \\
\uparrow & & \uparrow & & \uparrow \\
X^{\mathrm{tor}} & \longrightarrow & \mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) & \longrightarrow & \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor}})
\end{array}
$$

commutes. This is almost what we need. However, $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor}})$ is not involved in the construction from [GT], so we must check that invoking it is harmless. The diagram

$$
\begin{array}{ccc}
\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor}}) & \xrightarrow{\;p\;} & \mathrm{LCM}(\overline{X}^{\mathrm{red}}, X^{\mathrm{tor}}) \\
\downarrow {\scriptstyle \alpha} & & \downarrow {\scriptstyle \beta} \\
\overline{X}^{\mathrm{exc}} & \xrightarrow{\;\overline{q}\;} & \overline{X}^{\mathrm{red}}
\end{array}
$$

commutes: $\overline{q}\alpha = \beta p$. Let η be a homotopy inverse of the projection p. Then $\overline{q}\alpha\eta = \beta p\eta = \beta$, as desired. $\square$

The natural setting for Conjecture A is really the excentric compactifications. Indeed, our investigation of these in [HZ2:§1] was guided by the feeling that the two had much in common, yet they were not isomorphic as compactifications of X. We emphasize this by formulating the excentric version of Conjecture A:

Conjecture A′. *Let $k : X^{\mathrm{tor,exc}} \to \overline{X}^{\mathrm{exc}}$ be any of the continuous mappings constructed as above. Then the canonical extension $\mathcal{E}^{\mathrm{tor,exc}}$ is isomorphic to the pullback $k^* \overline{\mathcal{E}}^{\mathrm{exc}}$.*

(4.5) Proof of the excentric Goresky-Tai conjecture. We abstract the set-up just a little. Let Y_1 and Y_2 be two compactifications of X, with projections $p_1 : \mathrm{LCM}(Y_1, Y_2) \to Y_1$ and $p_2 : \mathrm{LCM}(Y_1, Y_2) \to Y_2$. Suppose that p_2 is a homotopy equivalence, with homotopy inverse $i : Y_2 \to \mathrm{LCM}(Y_1, Y_2)$. When X is a locally symmetric variety, we will be taking $Y_1 = \overline{X}^{\mathrm{exc}}$ and $Y_2 = X^{\mathrm{tor,exc}}$.

Suppose that for a vector bundle E on X, E has extensions $E_1 \to Y_1$ and $E_2 \to Y_2$. We assert:

Lemma 4.5.1. *In the above setting, put $k = p_1 \circ i$. Then the following are equivalent:*

i) $k^* E_1 \simeq E_2$,

ii) $p_1^* E_1 \simeq p_2^* E_2$ (as vector bundles on $\mathrm{LCM}(Y_1, Y_2)$).

Proof. Write i) out as $i^* p_1^* E_1 \simeq E_2$, and apply p_2^* to both sides. As $i \circ p_2$ is homotopic to the identity, we have $p_2^* i^*$ is the identity on isomorphism classes of vector bundles, giving ii). The other direction is similar: apply i^* to ii). $\square$

Thus, we seek a method for verifying $(4.5.1, \mathrm{ii})$ in the situation of interest, where we have $E_1 = \overline{\mathcal{E}}^{\mathrm{exc}}$ and $E_2 = \mathcal{E}^{\mathrm{tor,exc}}$. We continue, though, in the abstract setting.

Proposition 4.5.2. *Let $Y_1, Y_2 \in \mathfrak{PCp}(X)$, $E \to X$ a vector bundle, and $E_1 \to Y_1, E_2 \to Y_2$ vector bundle extensions of E. Concerning the diagram*

$$
\begin{array}{ccccc}
E_1 & \xleftarrow{\;\widetilde{p}_1\;} & \mathrm{LCM}(E_1, E_2) & \xrightarrow{\;\widetilde{p}_2\;} & E_2 \\
\varphi_1 \downarrow & & \downarrow \varphi & & \downarrow \varphi_2 \\
Y_1 & \xleftarrow{\;p_1\;} & \mathrm{LCM}(Y_1, Y_2) & \xrightarrow{\;p_2\;} & Y_2 ,
\end{array}
$$

the following are equivalent:

i) φ is a vector bundle projection;

ii) $\mathrm{LCM}(E_1, E_2) \simeq p_1^* E_1 \simeq p_2^* E_2$.

Proof. We show that i) implies ii), the other direction being obvious. By the universal property of a pullback, there is a canonical morphism $\mathrm{LCM}(E_1, E_2) \to p_1^* E_1$ over $\mathrm{LCM}(Y_1, Y_2)$. By assumption, it is a morphism of total spaces of vector

bundles of the same rank (that of E). To see it is an isomorphism, we may compute locally on $Y_1 \times Y_2$ and thereby assume that E_1 and E_2 are trivial. Thus, we write $E_1 \simeq Y_1 \times V$ and $E_2 \simeq Y_2 \times V'$, where $V = V'$ is a vector space. Then:

$$E_1 \times E_2 = (Y_1 \times V) \times (Y_2 \times V') \simeq (Y_1 \times Y_2) \times (V \times V'),$$
$$\mathrm{LCM}(E_1 \times E_2) \subset \mathrm{LCM}(Y_1, Y_2) \times (V \times V'),$$
$$p_1^* E_1 = \mathrm{LCM}(Y_1, Y_2) \times (V \times \{0\}).$$

The morphism $\mathrm{LCM}(E_1, E_2) \to p_1^* E_1$ is, in these terms, induced by the projection $\mathrm{LCM}(Y_1, Y_2) \times (V \times V') \to \mathrm{LCM}(Y_1, Y_2) \times (V \times \{0\})$. We have a priori that this contains as $E \to X$ as a subset, thus isomorphisms $\iota_x : V \to V'$ for $x \in X$. Our hypothesis implies the extension of $\{\iota_x\}$, at least as mappings, to all of $\mathrm{LCM}(Y_1, Y_2)$. Making the same argument for $p_2^* E_2$, we see that the extension of ι_x is invertible and ii) holds. $\square$

(4.5.2.1) *Remark.* It is instructive to consider the case $Y_1 = Y_2$ with E_1 and E_2 *non*-isomorphic.

With the criterion given by Proposition 4.5.2, it remains to verify:

Proposition 4.5.3. *The mapping* $\mathrm{LCM}(\overline{\mathcal{E}}^{\mathrm{exc}}, \mathcal{E}^{\mathrm{tor,exc}}) \to \mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}})$ *is a vector bundle projection.*

Proof. We must look simultaneously at the Borel-Serre and toroidal parametrizations of $\mathcal{E}$, relative to D_P. (In particular, we do not have to deal with the K_h-equivariance.) There is a K_ℓ-equivariant mapping

(4.5.3.1) $\qquad \Psi_P : G_{\ell,P} \times A_P \times U_P \times V_P \times E \to U_P^{\mathrm{im}} \times U_P \times V_P \times E,$

given by $\Psi_P(g, u, v, e) = (gu_0'g^{-1}, u, v, e)$, where u_0' is an element of C_P fixed by K_ℓ. This ultimately induces the pair of spaces: a $(T_P^c \times E)$-fibration over an arithmetic quotient of $(\overline{C}_P \times V_P)$, and a $(T_{P,\Sigma_P})^{\mathrm{exc}} \times E$-fibration over an arithmetic quotient of V_P. The LCM in the case of $E = \{0\}$ (for $\overline{X}^{\mathrm{exc}}$ and $X^{\mathrm{tor,exc}}$) is covered by Proposition 3.5.6. Since (4.5.3.1) is a product with the identity mapping of E, we see that the action of $\Gamma(G_{\ell,P} \cdot V_P)$ is diagonal for general $\mathcal{E}$, and $\mathrm{LCM}(\overline{\mathcal{E}}^{\mathrm{exc}}, \mathcal{E}^{\mathrm{tor,exc}})$ an E-fibration over $\mathrm{LCM}(\overline{X}^{\mathrm{exc}}, X^{\mathrm{tor,exc}})$, as we wanted to show. $\square$

By Theorem 4.4.1, the conjecture of Goresky and Tai (Conjecture A) is likewise proved.

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